Opening Exercises

If you have any tape diagramming experience, try to solve one of these problems using tape diagrams. If not, try to solve it algebraically.

94 children are in a reading club. One-third of the boys and three-sevenths of the girls prefer fiction. If 36 students prefer fiction, how many girls prefer fiction?

Jess spent one-third of her money on a cell phone, and two-fifths of the remainder on accessories. When she got home her parents gave her $350. The ratio of money she had in the end to the money she had before was 4:3. How much money did she have at first?

Example 1:

David spent $\frac{2}{5}$ of his money on a storybook. The storybook cost $\$20$. How much did he have at first?

Example 2:

Max spent $\frac{3}{5}$ of his money in a shop and $\frac{1}{4}$ of the remainder in another shop. What fraction of his money was left? If he had $\$90$ left, how much did he have at first?

Grade 6, Module 1, Lesson 1: Ratios

Student Outcomes

- Students understand that a *ratio* is an ordered pair of non-negative numbers, which are not both zero. Students understand that a ratio is often used instead of describing the first number as a multiple of the second.
- Students use the precise language and notation of ratios (e.g., $3:2$, 3 to 2). Students understand that the order of the pair of numbers in a ratio matters and that the description of the ratio relationship determines the correct order of the numbers. Students conceive of real-world contextual situations to match a given ratio.

Lesson Notes

The first two lessons of this module will develop the students' understanding of the term *ratio*. A ratio is always a pair of numbers, such as $2:3$ and never a pair of quantities such as 2 cm : 3 sec. Keeping this straight for students will require teachers to use the term ratio correctly and consistently. Students will be required to separately keep track of the units in a word problem. To help distinguish between ratios and statements about quantities that define ratios, we use the term *ratio relationship* to describe a phrase in a word problem that indicates a ratio. Typical examples of ratio relationship descriptions include "3 cups to 4 cups," "5 miles in 4 hours," etc. The ratios for these ratio relationships are $3:4$ and $5:4$, respectively.

Classwork

Example 1 (15 minutes)

Read the example aloud.

Example 1

The coed soccer team has four times as many boys on it as it has girls. We say the ratio of the number of boys to the number of girls on the team is $4:1$. We read this as "four to one."

- Let's create a table to show how many boys and how many girls could be on the team.

Create a table like the one shown below to show possibilities of the number of boys and girls on the soccer team. Have students copy the table into their student packet.

# of Boys	# of Girls	Total # of Players
4	1	5

- So, we would have four boys and one girl on the team for a total of five players. Is this big enough for a team?
  - *Adult teams require 11 players, but youth teams may have fewer. There is no right or wrong answer; just encourage the reflection on the question, thereby connecting their math work back to the context.*
- What are some other ratios that show four times as many boys as girls, or a ratio of boys to girls of 4 to 1?
  - *Have students add each ratio to their table.*

# of Boys	# of Girls	Total # of Players
4	1	5
8	2	10
12	3	15

- From the table, we can see that there are four boys for every one girl on the team.

Read the example aloud.

Suppose the ratio of number of boys to number of girls on the team is $3:2$.

Create a table like the one shown below to show possibilities of the number of boys and girls on the soccer team. Have students copy the table into their student materials.

# of Boys	# of Girls	Total # of Players
3	2	5

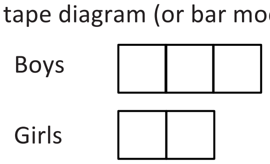
- What are some other team compositions where there are three boys for every two girls on the team?

# of Boys	# of Girls	Total # of Players
3	2	5
6	4	10
9	6	15

- I can't say there are 3 times as many boys as girls. What would my multiplicative value have to be? There are ____ as many boys as girls.

Encourage the students to articulate their thoughts, guiding them to say there are $\frac{3}{2}$ as many boys as girls.

- Can you visualize $\frac{3}{2}$ as many boys as girls?
- Can we make a tape diagram (or bar model) that shows that there are $\frac{3}{2}$ as many boys as girls?



- Which description makes the relationship easier to visualize: saying the ratio is 3 to 2 or saying there are 3 halves as many boys as girls?
  - *There is no right or wrong answer. Have students explain why they picked their choices.*

Example 2 (8 minutes): Class Ratios

Discussion (4 minutes)

Direct students:

- Find the ratio of boys to girls in our class.
- Raise your hand when you know: What is the ratio of boys to girls in our class?
- How can we say this as a multiplicative comparison without using ratios? Raise your hand when you know.

Allow for choral response when all hands are raised.

- Write the ratio of number of boys to number of girls in your student materials under Example 2, Question 1.
- Compare your answer with your neighbor's answer. Does everyone's ratio look exactly the same?

Allow for discussion of differences in what students wrote. Communicate the following in the discussions:

1. It is ok to use either the colon symbol or the word "to" between the two numbers of the ratio.
2. The ratio itself does not have units or descriptive words attached.

- Raise your hand when you know: What is the ratio of number of girls to number of boys in our class?
- Write the ratio down in your materials as number 2.
- Is the ratio of number of girls to number of boys the same as the ratio of number of boys to number of girls?
  - *Unless in this case there happens to be an equal number of boys and girls, then no, the ratios are not the same. Indicate that order matters.*
- Is this an interesting multiplicative comparison for this class? Is it worth commenting on in our class? If our class had 15 boys and 5 girls, might it be a more interesting observation?

For the exercise below, choose a way for students to indicate that they identify with the first statement (e.g., standing up or raising a hand). After each pair of statements below, have students create a ratio of the number of students who answered "yes" to the first statement to the number of students who answered "yes" to the second statement verbally, in writing, or both. Consider following each pair of statements with a discussion of whether it seems like an interesting ratio to discuss. Or alternatively, when you have finished all of these examples, ask students which ratio they found most interesting.

Students record a ratio for each of the examples you provide:

1. You traveled out of state this summer.
2. You did not travel out of state this summer.
3. You have at least one sibling.
4. You are an only child.
5. Your favorite class is math.
6. Your favorite class is not math.

Example 3:

Ingrid is mixing yellow and green paint together for a large art project. She uses a ratio of 2 pints of yellow paint for every 3 pints of green paint. Make up a problem for this situation.

Example 4:

Lena finds two boxes of printer paper in the teacher supply room. The ratio of the packs of paper in Box A to the packs of paper in Box B is $4:3$. If half of the paper in Box A is moved to Box B, what is the new ratio of packs of paper in Box A to Box B.

Example 5:

Sana and Amy collect bottle caps. The ratio of the number of bottle caps Sana has to the number Amy has is $2:3$. The ratio became $5:6$ when Sana added 8 more bottle caps to her collection. How many bottle caps does Amy have?

Example 6:

The ratio of songs on Jessa's phone to songs on Tessie's phone is 2 to 3. Tessie deletes half of her songs and now has 60 fewer songs than Jessa. How many songs does Jessa have?

Example 7:

Jack and Matteo had an equal amount of money each. After Jack spent $\$38$ and Matteo spent $\$32$, the ratio of Jack's money to Matteo's money was $3:5$. How much did each boy have at first?

Example 8:

The ratio of the number of Ingrid's stamps to the number of Ray's stamps is $3:7$. If Ingrid gives one-sixth of her stamps to Ray, what will be the new ratio of the number of Ingrid's stamps to the number of Ray's stamps?

Double Number Line Diagrams

Example: Rate Problems

A photocopier can print 12 copies in 36 seconds. At this rate, how many copies can it print in 1 minute?

Tape Diagrams vs. Double Number Line Diagrams

Monique walks 3 miles in 25 minutes.

vs.

Sean spends 5 minutes watching television for every 2 minutes he spends on homework.

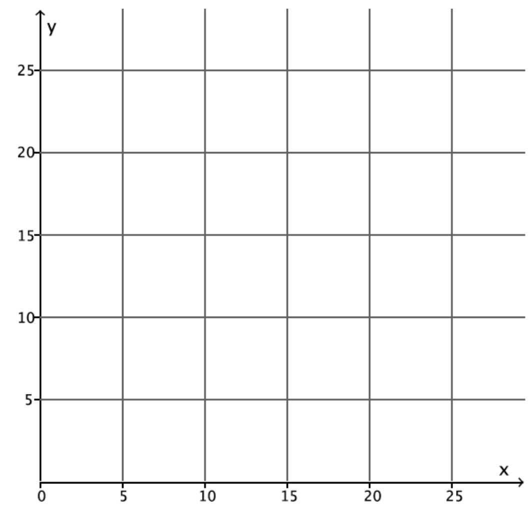
Ratio & Proportions

Exercise 1

In Jasmine's favorite fruit salad, the ratio of the number of cups of grapes to number of cups of peaches is 5 to 2.

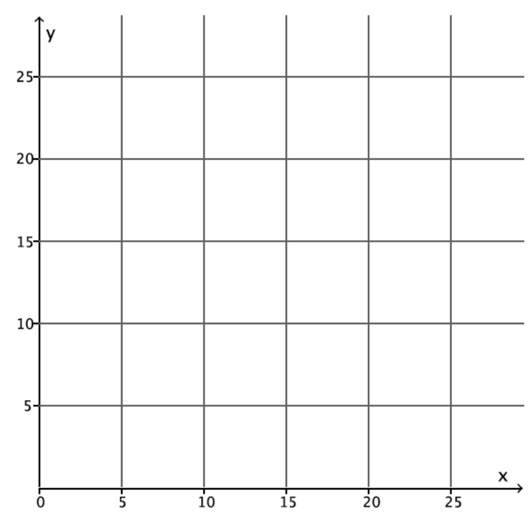
a. How many cups of peaches will be used if 25 cups of grapes are used?

b. Create both a ratio table and a graph of the proportional relationship to depict the relationship and find your answer.



c. What is the constant of proportionality?

d. Write the equation of the line depicted in the graph.

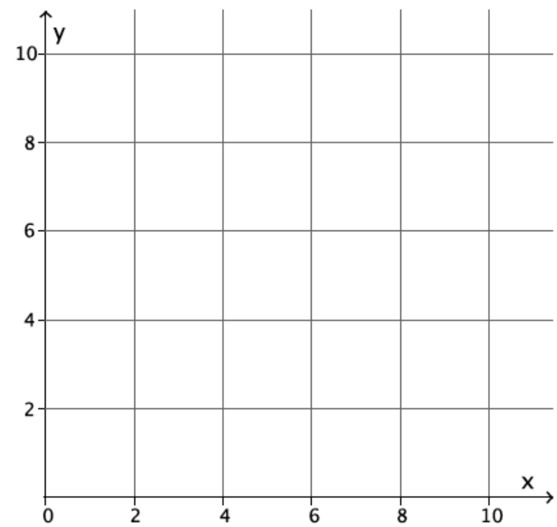


Exercise 2

Jack is taking a hike through a forested park. He moves at a constant rate, covering 5 miles every 2 hours.

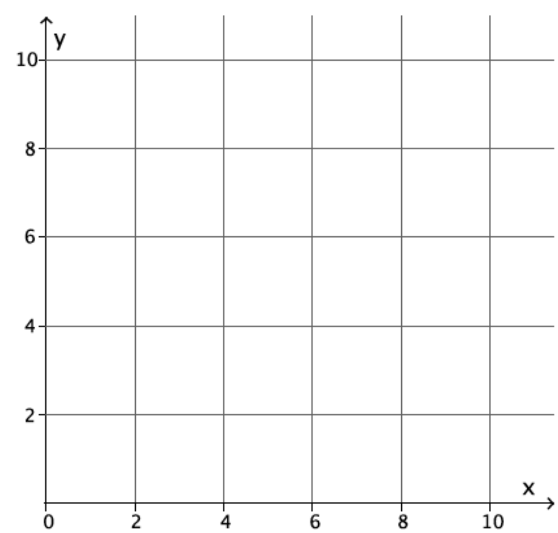
a. How much time will have passed when he has hiked 9 miles?

b. Create both a ratio table and a graph of the proportional relationship to depict the relationship and find your answer.



c. What is the constant of proportionality?

d. Write the equation of the line depicted in the graph.



Progressions: Ratios and Proportional Relationships (page 3)

Standards use ratio in the second sense, applying it to situations in which units are the same as well as to situations in which units are different. Relationships of two quantities in such situations may be described in terms of ratios, rates, percents, or proportional relationships.

A ratio associates two or more quantities. Ratios can be indicated in words as "3 to 2" and "3 for every 2" and "3 out of every 5" and "3 parts to 2 parts." This use might include units, e.g., "3 cups of flour for every 2 eggs" or "3 meters in 2 seconds." Notation for ratios can include the use of a colon, as in $3:2$. The quotient $\frac{3}{2}$ is sometimes called the value of the ratio $3:2$.•

Ratios have associated rates. For example, the ratio 3 feet for every 2 seconds has the associated rate $\frac{3}{2}$ feet for every 1 second; the ratio 3 cups apple juice for every 2 cups grape juice has the associated rate $\frac{3}{2}$ cups apple juice for every 1 cup grape juice. In Grades 6 and 7, students describe rates in terms such as "for each 1," "for each," and "per." The *unit rate* is the numerical part of the rate; the "unit" in "unit rate" is often used to highlight the 1 in "for each 1" or "for every 1."

Equivalent ratios arise by multiplying each measurement in a ratio pair by the same positive number. For example, the pairs of numbers of meters and seconds in the margin are in equivalent ratios. Such pairs are also said to be in the same ratio. Proportional relationships involve collections of pairs of measurements in equivalent ratios. In contrast, a proportion is an equation stating that two ratios are equivalent. Equivalent ratios have the same unit rate.

The pairs of meters and seconds in the margin show distance and elapsed time varying together in a *proportional relationship*. This situation can be described as "distance traveled and time elapsed are proportionally related," or "distance and time are directly proportional," or simply "distance and time are proportional." The proportional relationship can be represented with the equation $d = \left(\frac{3}{2}\right) t$. The factor $\frac{3}{2}$ is the constant unit rate associated with the different pairs of measurements in the proportional relationship; it is known as a *constant of proportionality*.

The word *percent* means "per 100" (*cent* is an abbreviation of the Latin *centum* "hundred"). If 35 milliliters out of every 100 milliliters in a juice mixture are orange juice, then the juice mixture is 35% orange juice (by volume). If a juice mixture is viewed as made of 100 equal parts, of which 35 are orange juice, then the juice mixture is 35% orange juice.

More precise definitions of the terms presented here and a framework for organizing and relating the concepts are presented in the Appendix.

• In everyday language. the word "ratio" sometimes refers to the value of a ratio, for example in the phrases "take the ratio of price to earnings" or "the ratio of circumference to diameter is π."

Representing pairs in a proportional relationship

Sharoya walks 3 meters every 2 seconds. Let d be the number of meters Sharoya has walked after t seconds. d and t are in a proportional relationship.

d meters	3	6	9	12	15	$\frac{3}{2}$	1	2	4
t seconds	2	4	6	8	10	1	$\frac{2}{3}$	$\frac{4}{3}$	$\frac{8}{3}$

d and t are related by the equation $d = \left(\frac{3}{2}\right) t$. Students sometimes use the equals sign incorrectly to indicate proportional relationships, for example, they might write "3 m = 2 sec" to represent the correspondence between 3 meters and 2 seconds. In fact, 3 meters is not equal to 2 seconds. This relationship can be represented in a table or by writing "3 m → 2 sec." Note that the unit rate appears in the pair $\left(\frac{3}{2}, 1\right)$.

Progressions: Ratios and Proportional Relationships (page 14)

A *proportional relationship* is a collection of pairs of numbers that are in equivalent ratios. A ratio $A : B$ determines a proportional relationship, namely the collection of pairs (cA, cB), for c positive. A proportional relationship is described by an equation of the form $y = kx$, where k is a positive constant, often called a *constant of proportionality*. The constant of proportionality, k, is equal to the value $\frac{B}{A}$. The graph of a proportional relationship lies on a ray with endpoint at the origin.

Two perspectives on ratios and their associated rates in quantitative contexts

Although ratios, rates, and proportional relationships can be described in purely numerical terms, these concepts are most often used with quantities.

Ratios are often described as comparisons by division, especially when focusing on an associated rate or value of the ratio. There are also two broad categories of basic ratio situations. Some division situations, notably those involving area, can fit into either category of division. Many ratio situations can be viewed profitably from within either category of ratio. For this reason, the two categories for ratio will be described as *perspectives* on ratio.

First perspective: Ratio as a composed unit or batch Two quantities are in a ratio of A to B if for every A units present of the first quantity there are B units present of the second quantity. In other words, two quantities are in a ratio of A to B if there is a positive number c (which could be a rational number), such that there are $c \cdot A$ units of the first quantity and $c \cdot B$ units of the second quantity. With this perspective, the two quantities can have the same or different units.

With this perspective, a ratio is specified by a composed unit or "batch," such as "3 feet in 2 seconds," and the unit or batch can be repeated or subdivided to create new pairs of amounts that are in the same ratio. For example, 12 feet in 8 seconds is in the ratio 3 to 2 because for every 3 feet, there are 2 seconds. Also, 12 feet in 8 seconds can be viewed as a 4 repetitions of the unit "3 feet in 2 seconds." Similarly, $\frac{3}{2}$ feet in 1 second is $\frac{1}{2}$ of the unit "3 feet in 2 seconds."

With this perspective, quantities that are in a ratio A to B give rise to a rate of $\frac{A}{B}$ units of the first quantity for every 1 unit of the second quantity (as well as to the rate of $\frac{B}{A}$ units of the second quantity for every 1 unit of the first quantity). For example, the ratio 3 feet in 2 seconds gives rise to the rate $\frac{3}{2}$ feet for every 1 second.

Two perspectives on ratio

1) There are 3 cups of apple juice for every 2 cups of grape juice in the mixture.

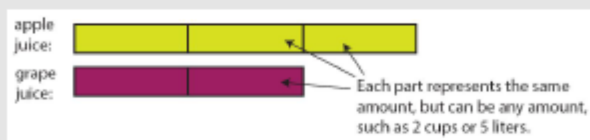
This way uses a composed unit: 3 cups apple juice and 2 cups grape juice. Any mixture that is made from some number of the composed unit is in the ratio 3 to 2.

In each of these mixtures, apple juice and grape juice are mixed in a ratio of 3 to 2:

# cups apple juice	3	6	9	12	3/2	1
# cups grape juice	2	4	6	8	1	2/3

made of 2 composed units — made of 1/2 of a composed unit

2) The mixture is made from 3 parts apple juice and 2 parts grape juice, where all parts are the same size, but can be any size.



If 1 part is :	1 cup	2 cups	5 liters	3 quarts
amt of apple juice:	3 cups	6 cups	15 liters	9 quarts
amt of grape juice:	2 cups	4 cups	10 liters	6 quarts

Progressions: Ratios and Proportional Relationships (pages 8–9)

Grade 7

In Grade 7, students extend their reasoning about ratios and proportional relationships in several ways. Students use ratios in cases that involve pairs of rational number entries, and they compute associated unit rates. They identify these unit rates in representations of proportional relationships. They work with equations in two variables to represent and analyze proportional relationships. They also solve multi-step ratio and percent problems, such as problems involving percent increase and decrease.

At this grade, students will also work with ratios specified by rational numbers, such as $\frac{3}{4}$ cups flour for every $\frac{1}{2}$ stick butter.[7.RP.1] Students continue to use ratio tables, extending this use to finding unit rates.

Recognizing proportional relationships Students examine situations carefully, to determine if they describe a proportional relationship.[7.RP.2a] For example, if Josh is 10 and Reina is 7, how old will Reina be when Josh is 20? We cannot solve this problem with the proportion $\frac{10}{7} = \frac{20}{R}$ because it is not the case that for every 10 years that Josh ages, Reina ages 7 years. Instead, when Josh has aged 10 another years, Reina will as well, and so she will be 17 when Josh is 20.

For example, if it takes 2 people 5 hours to paint a fence, how long will it take 4 people to paint a fence of the same size (assuming all the people work at the same steady rate)? We cannot solve this problem with the proportion $\frac{2}{5} = \frac{4}{H}$ because it is not the case that for every 2 people, 5 hours of work are needed to paint the fence. When more people work, it will take fewer hours. With twice as many people working, it will take half as long, so it will take only 2.5 hours for 4 people to paint a fence. Students must understand the structure of the problem, which includes looking for and understand the roles of "for every," "for each," and "per."

Students recognize that graphs that are not lines through the origin and tables in which there is not a constant ratio in the entries do not represent proportional relationships. For example, consider circular patios that could be made with a range of diameters. For such patios, the area (and therefore the number of pavers it takes to make the patio) is not proportionally related to the diameter, although the circumference (and therefore the length of stone border it takes to encircle the patio) is proportionally related to the diameter. Note that in the case of the circumference, C, of a circle of diameter D, the constant of proportionality in $C = \pi \cdot D$ is the number π, which is not a rational number.

Equations for proportional relationships As students work with proportional relationships, they write equations of the form $y = cx$, where c is a constant of proportionality, i.e., a unit rate.[7.RP.2c] They

7.RP.1 Compute unit rates associated with ratios of fractions, including ratios of lengths, areas and other quantities measured in like or different units.

Ratio problem specified by rational numbers: Three approaches

To make Perfect Purple paint mix $\frac{1}{2}$ cup blue paint with $\frac{1}{3}$ cup red paint. If you want to mix blue and red paint in the same ratio to make 20 cups of Perfect Purple paint, how many cups of blue paint and how many cups of red paint will you need?

Method 1

		$\cdot 6$	$\cdot 4$
cups blue	$\frac{1}{2}$	3	12
cups red	$\frac{1}{3}$	2	8
total cups purple	$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$	5	20

"I thought about making 6 batches of purple because that is a whole number of cups of purple. To make 6 batches, I need 6 times as much blue and 6 times as much red too. That was 3 cups blue and 2 cups red and that made 5 cups purple. Then 4 times as much of each makes 20 cups purple."

Method 2

$$\frac{1}{2} \div \frac{5}{6} = \frac{1}{2} \cdot \frac{6}{5} = \frac{6}{10} \qquad \frac{6}{10} \cdot 20 = 12$$

$$\frac{1}{3} \div \frac{5}{6} = \frac{1}{3} \cdot \frac{6}{5} = \frac{6}{15} \qquad \frac{6}{15} \cdot 20 = 8$$

"I found out what fraction of the paint is blue and what fraction is red. Then I found those fractions of 20 to find the number of cups of blue and red in 20 cups."

Method 3

cups blue	$\frac{1}{2}$	12	
cups red	$\frac{1}{3}$	8	
total cups purple	$\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$	20	$\cdot \frac{2}{5}$, $\cdot \frac{3}{5}$

Like Method 2, but in tabular form, and viewed as multiplicative comparisons.

7.RP.2a Recognize and represent proportional relationships between quantities.

a Decide whether two quantities are in a proportional relationship, e.g., by testing for equivalent ratios in a table or graphing on a coordinate plane and observing whether the graph is a straight line through the origin.

7.RP.2c Represent proportional relationships by equations.

see this unit rate as the amount of increase in y as x increases by 1 unit in a ratio table and they recognize the unit rate as the vertical increase in a "unit rate triangle" or "slope triangle" with horizontal side of length 1 for a graph of a proportional relationship.[7.RP.2b]

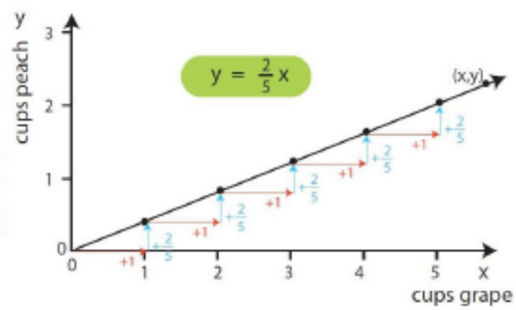
7.RP.2b Recognize and represent proportional relationships between quantities.

b Identify the constant of proportionality (unit rate) in tables, graphs, equations, diagrams, and verbal descriptions of proportional relationships.

Correspondence among a table, graph, and equation of a proportional relationship

For every 5 cups grape juice, mix in 2 cups peach juice.

x cups grape	y cups peach
(0)	(0)
5	2
1	$\frac{2}{5}$
2	$2 \cdot \frac{2}{5}$
3	$3 \cdot \frac{2}{5}$
4	$4 \cdot \frac{2}{5}$
x	$x \cdot \frac{2}{5}$



On the graph: For each 1 unit you move to the right, move up $\frac{2}{5}$ of a unit.

When you go 2 units to the right, you go up $2 \cdot \frac{2}{5}$ units.

When you go 3 units to the right, you go up $3 \cdot \frac{2}{5}$ units.

When you go 4 units to the right, you go up $4 \cdot \frac{2}{5}$ units.

When you go x units to the right, you go up $x \cdot \frac{2}{5}$ units.

Starting from $(0, 0)$, to get to a point (x, y) on the graph, go x units to the right, so go up $x \cdot \frac{2}{5}$ units.

Therefore $y = x \cdot \frac{2}{5}$ $\quad y = \frac{2}{5}x$

Students connect their work with equations to their work with tables and diagrams. For example, if Seth runs 5 meters every 2 seconds, then how long will it take Seth to run 100 meters at that rate? The traditional method is to formulate an equation, $\frac{5}{2} = \frac{100}{T}$, cross-multiply, and solve the resulting equation to solve the problem. If $\frac{5}{2}$ and $\frac{100}{T}$ are viewed as unit rates obtained from the equivalent ratios $5 : 2$ and $100 : T$, then they must be equivalent fractions because equivalent ratios have the same unit rate. To see the rationale for cross-multiplying, note that when the fractions are given the common denominator $2 \cdot T$, then the numerators become $5 \cdot T$ and $2 \cdot 100$ respectively. Once the denominators are equal, the fractions are equal exactly when their numerators are equal, so $5 \cdot T$ must equal $2 \cdot 100$ for the unit rates to be equal. This is why we can solve the equation $5 \cdot T = 2 \cdot 100$ to find the amount of time it will take for Seth to run 100 meters.

A common error in setting up proportions is placing numbers in incorrect locations. This is especially easy to do when the order in which quantities are stated in the problem is switched within the problem statement. For example, the second of the following two

Grade 8, Module 4, Lesson 10: A Critical Look at Proportional Relationships

Example 1:

- Consider the word problem below. We can do several things to answer this problem, but let's begin to organize our work using a table for time and distance:

  Paul walks 2 miles in 25 minutes. How many miles can Paul walk in 137.5 minutes?

Scaffolding:

It may be necessary to remind students of the relationship between distance traveled, rate, and time spent traveling at that rate.

Time (in minutes)	Distance (in miles)
25	2
50	4
75	6
100	8
125	10

As students answer the questions below, fill in the table.

- How many miles would Paul be able to walk in 50 minutes? Explain.
  - *Paul could walk 4 miles in 50 minutes because 50 minutes is twice the time we were given, so we can calculate twice the distance, which is 4.*
- How many miles would Paul be able to walk in 75 minutes? Explain.
  - *Paul could walk 6 miles in 75 minutes because 75 minutes is three times the number of minutes we were given, so we can calculate three times the distance, which is 6.*
- Since the relationship between the distance Paul walks and the time it takes him to walk that distance is proportional, we let y represent the distance Paul walks in 137.5 minutes and write:

$$\frac{25}{2} = \frac{137.5}{y}$$
$$25y = 137.5(2)$$
$$25y = 275$$
$$y = 11$$

Therefore, Paul can walk 11 miles in 137.5 minutes.

- How many miles y can Paul walk in x minutes?

Provide students time to think about the answer to this question. Allow them to share their ideas and then proceed with the discussion below, if necessary.

- We know for a fact that Paul can walk 2 miles in 25 minutes, so we can write the ratio $\frac{25}{2}$ as we did with the proportion. We can write another ratio for the number of miles y Paul walks in x minutes. It is $\frac{x}{y}$. For the same reason we could write the proportion before, we can write one now with these two ratios:

$$\frac{25}{2} = \frac{x}{y}$$

Example 2:

The point of this problem is to make clear to students that we must assume constant rate in order to write linear equations in two variables and to use those equations to answer questions about distance, time, and rate.

- Consider the following word problem: Alexxa walked from the Grand Central Station on 42^{nd} street to the Penn Station on 7^{th} avenue. The total distance traveled was 1.1 miles. It took Alexxa 25 minutes to make the walk. How many miles did she walk in the first 10 minutes?

Give students a minute to think and/or work on the problem. Expect them to write a proportion and solve the problem. The next part of the discussion will get them to think about what is meant by "constant" speed or rather lack of it.

  - *She walked* 0.44 *miles. (Assuming students used a proportion to solve.)*

- Are you sure about your answer? How often do you walk at a constant speed? Notice the problem did not even mention that she was walking at the same rate throughout the entire 1.1 miles. What if you have more information about her walk: Alexxa walked from Grand Central Station (GCS) along 42^{nd} street to an ATM machine 0.3 miles away in 8 minutes. It took her 2 minutes to get some money out of the machine. Do you think your answer is still correct?
  - *Probably not since now that we know she had to stop at the ATM.*
- Let's continue with Alexxa's walk: She reached the 7^{th} avenue junction 13 minutes after she left GCS, a distance of 0.6 miles. There she met her friend Karen with whom she talked for 2 minutes. After leaving her friend she finally got to Penn Station 25 minutes after her walk began.
- Isn't this a more realistic situation than believing that she walked the exact same speed throughout the entire trip? What other events typically occur during walks in the city?
  - *Stop lights at crosswalks, traffic, maybe a trip/fall, running an errand, etc.*
- This is precisely the reason we need to take a critical look at what we call "proportional relationships" and constant speed, in general.
- The following table shows an accurate picture of Alexxa's walk:

Time(in minutes)	Distance Traveled (in miles)
0	0
8	0.3
10	0.3
13	0.6
15	0.6
25	1.1

With this information, we can answer the question. Alexxa walked *exactly* 0.3 miles in 10 minutes.

- Now that we have an idea of what could go wrong when we assume a person walks at a constant rate or that a proportion can give us the correct answer all of the time, let's define what is called *average speed.*

  Suppose a person walks a distance of d (miles) in a given time interval t (minutes). Then the **average speed** in the given time interval is $\frac{d}{t}$ in miles per minute.

Grade 7, Module 1, Lesson 3: Identifying Proportional and Non-Proportional Relationships in Tables

Exercises 1-3

For Exercises 1–3, determine if y is proportional to x. Justify your answer.

1. The table below represents the relationship of the amount of snowfall (in inches) in 5 counties to the amount of time (in hours) hours of a recent winter storm.

x Time (h)	y Snowfall (in.)
2	10
6	12
8	16
2.5	5
7	14

2. The table below shows the relationship between the cost of renting a movie (in dollars) to the number of days the movie is rented.

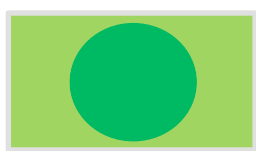
x Number of Days	y Cost (dollars)
6	2
9	3
24	8
3	1

3. The table below shows the relationship between the amount of candy bought (in pounds) and the total cost of the candy (in dollars).

x Amount of Candy (pounds)	y Cost (dollars)
5	10
4	8
6	12
8	16
10	20

Grade 7, Module 1, End-of-Module Assessment, Item 2

2. Farmers often plant crops in circular areas because one of the most efficient watering systems for crops provides water in a circular area. Passengers in airplanes often notice the distinct circular patterns as they fly over land used for farming. A photographer takes an aerial photo of a field on which a circular crop area has been planted. He prints the photo out and notes that 2 centimeters of length in the photo corresponds to 100 meters in actual length.



a. What is the scale factor of the actual farm to the photo?

b. If the dimensions of the entire photo are 25 cm by 20 cm, what are the actual dimensions of the rectangular land area in meters captured by the photo?

Grade 7, Module 4, Lesson 3: Comparing Quantities with Percent

Exit Ticket

Solve each problem below using at least two different approaches.

Jenny's great grandmother is 90 years old. Jenny is 12 years old. What percent of Jenny's great grandmother's age is Jenny's age?

Number Correct: _______

Part, Whole, or Percent – Round 1

Directions: Find each missing value.

1.	1% of 100 is?		23.	10% of 22 is?	
2.	2% of 100 is?		24.	20% of 22 is?	
3.	3% of 100 is?		25.	30% of 22 is?	
4.	4% of 100 is?		26.	50% of 22 is?	
5.	5% of 100 is?		27.	25% of 22 is?	
6.	9% of 100 is?		28.	75% of 22 is?	
7.	10% of 100 is?		29.	80% of 22 is?	
8.	10% of 200 is?		30.	85% of 22 is?	
9.	10% of 300 is?		31.	90% of 22 is?	
10.	10% of 500 is?		32.	95% of 22 is?	
11.	10% of 550 is?		33.	5% of 22 is?	
12.	10% of 570 is?		34.	15% of 80 is?	
13.	10% of 470 is?		35.	15% of 60 is?	
14.	10% of 170 is?		36.	15% of 40 is?	
15.	10% of 70 is?		37.	30% of 40 is?	
16.	10% of 40 is?		38.	30% of 70 is?	
17.	10% of 20 is?		39.	30% of 60 is?	
18.	10% of 25 is?		40.	45% of 80 is?	
19.	10% of 35 is?		41.	45% of 120 is?	
20.	10% of 36 is?		42.	120% of 40 is?	
21.	10% of 37 is?		43.	120% of 50 is?	
22.	10% of 37.5 is?		44.	120% of 55 is?	

Number Correct: ______

Improvement: ______

Part, Whole, or Percent – Round 2

Directions: Find each missing value.

1.	20% of 100 is?		23.	10% of 4 is?	
2.	21% of 100 is?		24.	20% of 4 is?	
3.	22% of 100 is?		25.	30% of 4 is?	
4.	23% of 100 is?		26.	50% of 4 is?	
5.	25% of 100 is?		27.	25% of 4 is?	
6.	25% of 200 is?		28.	75% of 4 is?	
7.	25% of 300 is?		29.	80% of 4 is?	
8.	25% of 400 is?		30.	85% of 4 is?	
9.	25% of 4000 is?		31.	90% of 4 is?	
10.	50% of 4000 is?		32.	95% of 4 is?	
11.	10% of 4000 is?		33.	5% of 4 is?	
12.	10% of 4700 is?		34.	15% of 40 is?	
13.	10% of 4600 is?		35.	15% of 30 is?	
14.	10% of 4630 is?		36.	15% of 20 is?	
15.	10% of 463 is?		37.	30% of 20 is?	
16.	10% of 46.3 is?		38.	30% of 50 is?	
17.	10% of 18 is?		39.	30% of 90 is?	
18.	10% of 24 is?		40.	45% of 90 is?	
19.	10% of 3.63 is?		41.	90% of 120 is?	
20.	10% of 0.336 is?		42.	125% of 40 is?	
21.	10% of 37 is?		43.	125% of 50 is?	
22.	10% of 37.5 is?		44.	120% of 60 is?	

Grade 7, Module 4, Lesson 4: Percent Increase and Decrease

Opening Exercise

Cassandra likes jewelry. She has five rings in her jewelry box.

a. In the box below, sketch Cassandra's five rings.

b. Draw a double number line diagram relating the number of rings as a percent of the whole set of rings.

c. What percent is represented by the whole collection of rings? What percent of the collection does each ring represent?

Exit Ticket

Erin wants to raise her math grade to a 95 to improve her chances of winning a math scholarship. Her math average for the last marking period was an 81. Erin decides she must raise her math average by 15% to meet her goal. Do you agree? Why or why not? Support your written answer by showing your math work.

Grade 6, Module 2, Lesson 1: Interpreting Division of a Fraction by a Whole Number – Visual Models

Example 1:

This lesson will focus on fractions divided by whole numbers. Students learned how to divide unit fractions by whole numbers in 5th grade. Teachers can become familiar with what was taught in 5th grade on this topic by reviewing the materials used in the Grade 5, Module 4 lessons and assessments.

Example 1

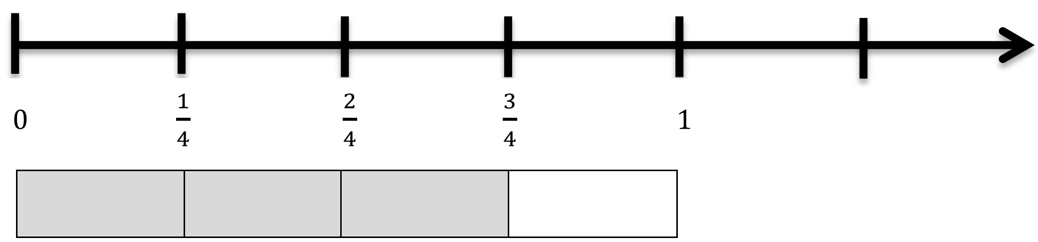
Maria has $\frac{3}{4}$ lb. of trail mix. She needs to share it equally among 6 friends. How much will each friend be given? What is this question asking us to do?

We are being asked to divide the trail mix into six equal portions. So we need to divide three-fourths by six.

How can this question be modeled?

- Let's take a look at how to solve this using a number line and a fraction bar.

We will start by creating a number line broken into fourths and a fraction bar broken into fourths.



Grade 6, Module 2, Lesson 2: Interpreting Division of a Fraction by a Whole Number – Visual Models

Example 1:

At the beginning of class, break students into groups. Each group will need to answer the question it has been assigned and draw a model to represent its answer. Multiple groups could have the same question.

Group 1: How many half-miles are in 12 miles? $12 \div \frac{1}{2} = 24$

Group 2: How many quarter hours are in 5 hours? $5 \div \frac{1}{4} = 20$

Group 3: How many one-third cups are in 9 cups? $9 \div \frac{1}{3} = 27$

MP.1 & MP.2

Group 4: How many one-eighth pizzas are in 4 pizzas? $4 \div \frac{1}{8} = 32$

Group 5: How many one-fifths are in 7 wholes? $7 \div \frac{1}{5} = 35$

Models will vary but could include fraction bars, number lines, or area models (arrays).

Students will draw models on blank paper, construction paper, or chart paper. Hang up only student models, and have students travel around the room answering the following:

4. Write the division question that was answered with each model.
5. What multiplication question could this model also answer?
6. Rewrite the question given to each group as a multiplication question.

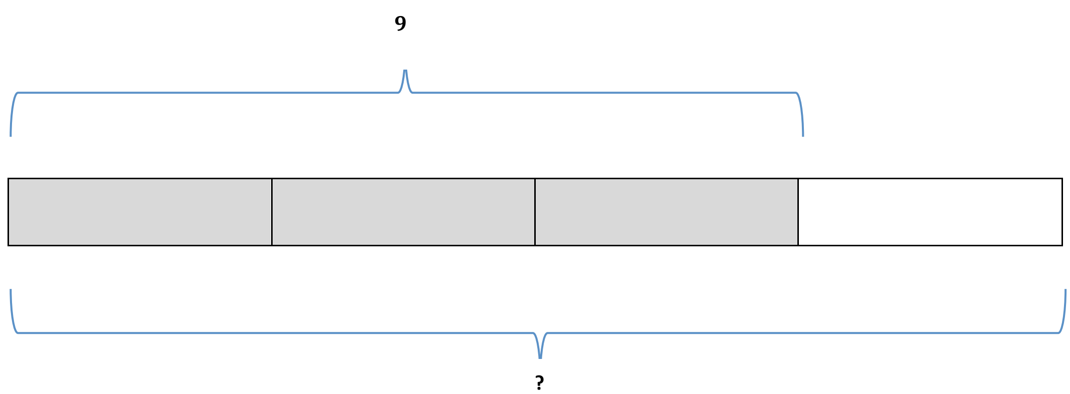
Students will be given a table to fill in as they visit each model.

Example 2:

Molly used 9 cups of flour to bake bread. If this was $\frac{3}{4}$ of the total amount of flour she started with, what was the original amount of flour?

- What is different about this question from the measurement questions?
  - *In this example, we are not trying to figure out how many three-fourths are in* 9*. We know that* 9 *cups is a part of the entire amount of flour needed. Instead, we need to determine three-fourths of what number is* 9*.*

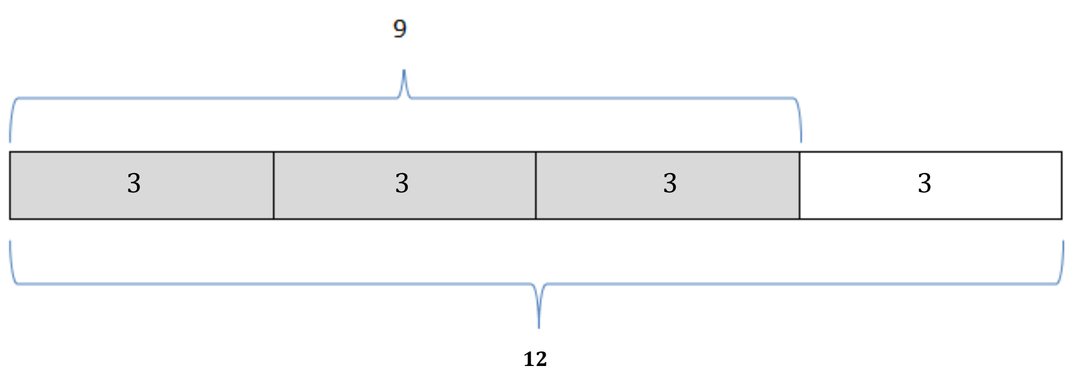
a. **Create a model to represent what the question is asking for.**



b. **Explain how you would determine the answer using the model.**

To divide 9 ***by*** $\frac{3}{4}$***, we can divide*** 9 ***by*** 3 ***to get the amount for each rectangle and then multiply by*** 4 ***because there are*** 4 ***rectangles total.***

$9 \div 3 = 3 \quad 3 \times 4 = 12$. ***Now, I can see that there were originally*** 12 ***cups of flour.***



Grade 6, Module 2, Lesson 3: Interpreting and Computing Division of a Fraction by a Fraction – More Models

Example 1:

Next, we will introduce an example where students are asked to divide a fraction by a fraction with the same denominator. The whole number examples in the opening are used to give students ideas to build off of when dealing with fractions.

- What is $\frac{8}{9} \div \frac{2}{9}$? Take a moment to use what you know about division to create a model to represent this division problem.

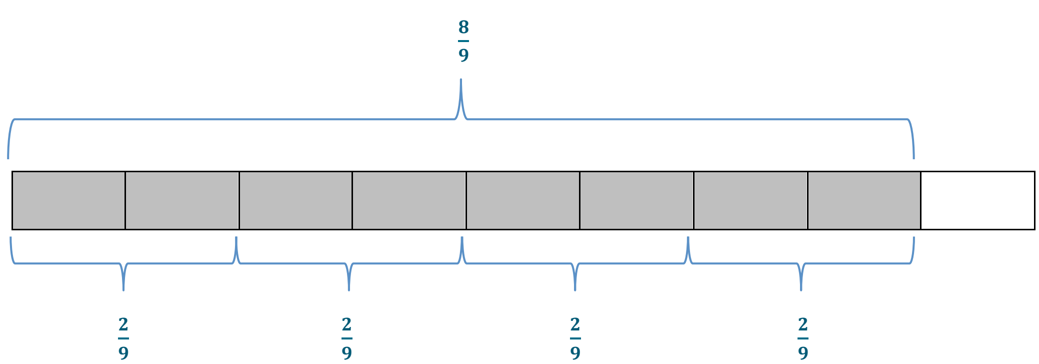
Give students a chance to explore this question and draw models without giving them the answer first. After three minutes or so, ask for students to share the models that they have created and to discuss what conclusions they have made about dividing fractions with the same denominator.

  - *One way to interpret the question is to say how many $\frac{2}{9}$ are in $\frac{8}{9}$. From the model, I can see that there are 4 groups of $\frac{2}{9}$ in $\frac{8}{9}$. This would give the same solution as dividing 8 by 2 to get 4.*

Example 1

$\frac{8}{9} \div \frac{2}{9}$

Draw a model to show the division problem.



Here we have 4 groups of $\frac{2}{9}$. Therefore, the answer is 4.

Equations and Expressions: A Progression of Each Capacity

1. Replace numbers in numerical expressions and number sentences with variable to express generalizations.

    G6-M4 Lessons 1-4, 8

2. Write algebraic expressions to represent a verbal expression (or statement of equality or inequality, and vice versa).

    a. Appropriately describe the number that the variable(s) represents.

    G6-M4 Lessons 9-10, 13-22, 34

    G7-M2 Lessons 18 and 19

    G8-M4 Lesson 1

3. Manipulate expressions using the properties of numbers and properties of operations.

    a. Be certain about whether two expressions are algebraically equivalent.

    b. Develop an intuition about what manipulations might be useful in a given situation.

    G6-M4 Lessons 5-6, 9-12

    G7-M3 Lessons 1-6

    G8-M4

4. Evaluate an expression by replacing the variables with a single number.

    G6-M4 Lessons 7, 18-22

    G7-M3 Lessons 16-26

    G8-M4 Lesson 12

5. Solving equations (or inequalities), that is, finding the value(s) of a variable that creates a true number sentence, given a statement of equality between two expressions.

    a. Using the properties of operations and numbers on either expression, and using the properties of equality.

    G6-M4 Lessons 23-24

    G7-M3 Lessons 7-15, some of 16-26

    G8-M4 Lessons 3-8, 10-14, 24-31

Grade 6, Module 4, Lesson 1: The Relationship of Addition and Subtraction

1. Write a number sentence, using variables, to represent the identities we demonstrated with tape diagrams.

2. Using your knowledge of identities, fill in each of the blanks.

   a. $4 + 5 -$ _____ $= 4$

   b. $25 -$ _____ $+ 10 = 25$

   c. ____ $+16 - 16 = 45$

   d. $56 - 20 + 20 =$_____

5. Using your knowledge of identities, fill in each of the blanks.

   a. $a + b -$ _____ $= a$

   b. $c - d + d =$ _____

   c. $e +$ _____ $-f = e$

   d. _____ $-h + h = g$

Grade 6, Module 4, Lesson 9: Writing Addition and Subtraction Expressions

Example 6

How would we write an expression to show the number c being subtracted from the sum of a and b?

- Start by writing an expression for "the sum of a and b."

- Now show c being subtracted from the sum.

Example 7

Write an expression to show the number c minus the sum of a and b.

- Why are the parentheses necessary in this example and not the others?

- Replace the variables with numbers to see if $c - (a + b)$ is the same as $c - a + b$.

Grade 8, Module 4, Lesson 1: Writing Equations Using Symbols

Exercises 1-4

Write each of the following statements using symbolic language.

1. The sum of four consecutive even integers is -28.

2. A number is four times larger than the square of half the number.

3. Steven has some money. If he spends nine dollars, then he will have $\frac{3}{5}$ of the amount he started with.

4. The sum of a number squared and three less than twice the number is 129.

Grade 6, Module 4, Lesson 10: Writing and Expanding Multiplication Expressions

Example 1

Write each expression using the fewest number of symbols and characters. Use math terms to describe the expressions and parts of the expression.

a. $6 \times b$

b. $4 \cdot 3 \cdot h$

c. $2 \times 2 \times 2 \times a \times b$

d. $5 \times m \times 3 \times p$

e. $1 \times g \times w$

Grade 6, Module 4, Lesson 20: Writing and Evaluating Expressions—Multiplication and Division

2. In New York State, there is a five-cent deposit on all carbonated beverage cans and bottles. When you return the empty can or bottle, you get the five cents back.

   f. Complete the table.

Number of Containers Returned	Refund in Dollars
1	
2	
3	
4	
10	
50	
100	
C	

   g. If we let C represent the number of cans, what is the expression that shows how much money is returned?

   h. Use the expression to find out how much money Brett would receive if he returned 222 cans.

   i. If Gavin needs to earn $\$4.50$ for returning cans, how many cans does he need to collect and return?

   j. How is part (d) different from part (c)?

Grade 7, Module 3, Lesson 1: Generating Equivalent Expressions

Problem Set

For problems 1–9, write equivalent expressions by combining like terms. Verify the equivalence of your expression and the given expression by evaluating each for the given values: $a = 2$, $b = 5$, and $c = -3$.

1. $3a + 5a$
2. $8b - 4b$
3. $5c + 4c + c$
4. $3a + 6 + 5$
5. $8b + 8 - 4b$
6. $5c - 4c + c$
7. $3a + 6 + 5a - 2$
8. $8b + 8 - 4b - 3$
9. $5c - 4c + c - 3c$

Use any order, any grouping to write equivalent expressions by combining like terms. Then verify the equivalence of your expression to the given expression by evaluating for the value(s) given in each problem.

10. $3(6a)$; for $a = 3$
11. $5d(4)$; for $d = -2$
12. $(5r)(-2)$; for $r = -3$
13. $3b(8) + (-2)(7c)$; for $b = 2, c = 3$
14. $-4(3s) + 2(-t)$; for $s = \frac{1}{2}$, $t = -3$
15. $9(4p) - 2(3q) + p$; for $p = -1, q = 4$
16. $7(4g) + 3(5h) + 2(-3g)$; $g = \frac{1}{2}, h = \frac{1}{3}$

Grade 6, Module 4, Lesson 26: One-Step Equations—Addition and Subtraction

Exercise 1

Solve each equation. Use both tape diagrams and algebraic methods for each problem. Use substitution to check your answers.

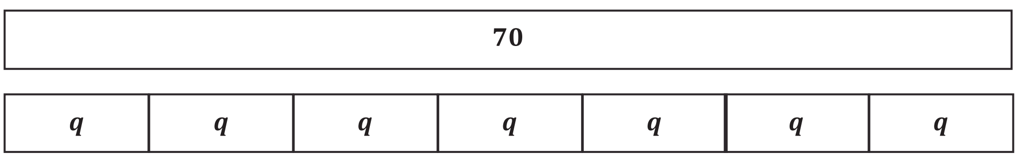
a. $b + 9 = 15$

b. $12 = 8 + c$

Grade 6, Module 4, Lesson 27: One-Step Equations—Multiplication and Division

3. Calculate the solution of the equation using the method of your choice: $4p = 36$.

17. Examine the tape diagram below and write an equation it represents. Then, calculate the solution to the equation using the method of your choice.

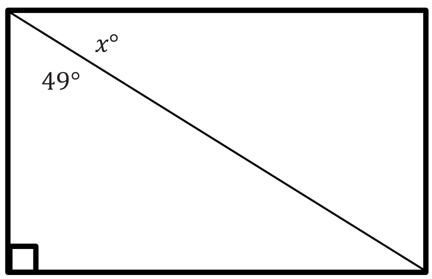


18. Write a multiplication equation that has a solution of 12. Use tape diagrams to prove that your equation has a solution of 12.

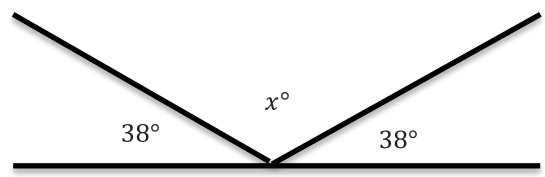
19. Write a division equation that has a solution of 12. Prove that your equation has a solution of 12 using algebraic methods.

Grade 6, Module 4, Lesson 30: One-Step Problems in the Real World

3. Candice is building a rectangular piece of fence according to the plans her boss gave her. One of the angles is not labeled. Write an equation and use it to determine the measure of the unknown angle.



4. Rashid hit a hockey puck against the wall at a $38°$ angle. The puck hit the wall and traveled in a new direction. Determine the missing angle in the diagram.



5. Jaxon is creating a mosaic design on a rectangular table. He has added two pieces to one of the corners. The first piece has an angle measuring $38°$ that is placed in the corner. A second piece has an angle measuring $27°$ that is also placed in the corner. Draw a diagram to model the situation. Then, write an equation and use it to determine the measure of the unknown angle in a third piece that could be added to the corner of the table.

Grade 7, Module 3, Lesson 7: Understanding Equations

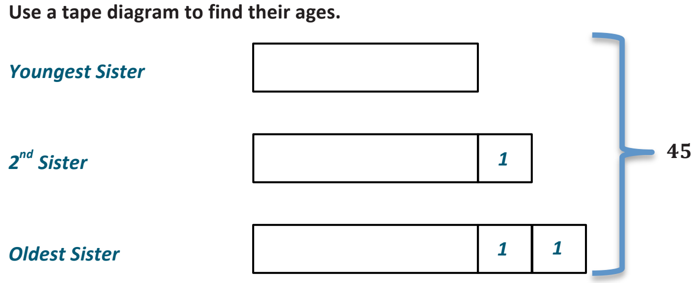
- Does it matter which equation you use when determining which given values make the equation true? Explain how you know.
  - *Yes, since the values were given in inches, the equation* $3y + 25 + 25 = 128$ *can be used because each term of the equation is in the same unit of measure.*
- If one uses the other equation, what must be done to obtain the solution?
  - *If the other equation were used, then the given values of* 24, 25, *and* 26 *inches need to be converted to* 2, $2\frac{1}{12}$, *and* $2\frac{1}{6}$ *feet, respectively.*

Example 1:

The example is a consecutive integer word problem. A tape diagram is used to model an arithmetic solution in part (a). Replacing the first bar (the youngest sister's age) in the tape diagram with x years provides an opportunity for students to visualize the meaning of the equation created in part (b).

The ages of three sisters are consecutive integers. The sum of their ages is 45. Find their ages.

a. Use a tape diagram to find their ages.



$45 - 3 = 42.$

$42 \div 3 = 14.$

Youngest Sister: 14 *years old*

2nd Sister: 15 *years old*

Oldest Sister: 16 *years old*

b. If the youngest sister is x years old, describe the ages of the other two sisters in terms of x, write an expression for the sum of their ages in terms of x, and use that expression to write an equation that can be used to find their ages.

Youngest Sister: x *years old*

2nd Sister: $(x+1)$ *years old*

Oldest Sister: $(x+2)$ *years old*

Sum of their ages: $x + (x+1) + (x+2)$

Equation: $x + (x+1) + (x+2) = 45$

c. **Determine if your answer from part (a) is a solution to the equation you wrote in part (b).**

$$\mathbf{x + (x+1) + (x+2) = 45}$$
$$\mathbf{14 + (14+1) + (14+2) = 45}$$

True

Scaffolding:

Review what is meant by consecutive integers–positive and negative whole numbers that increase or decrease by 1 unit. For example: $-2, -1, 0$.

- Let x be an integer; write an algebraic expression that represents one more than that integer.
  - $x + 1$
- Write an algebraic expression that represents two more than that integer.
  - $x + 2$

Discuss how the unknown unit in a tape diagram represents the unknown integer, represented by x. Consecutive integers begin with the unknown unit; then, every consecutive integer thereafter increases by 1 unit.

Exercise

Instruct students to complete the following exercise individually and discuss the solution as a class.

1. **Sophia pays a $\$19.99$ membership fee for an online music store.**

   a. **If she also buys two songs from a new album at a price of $\$0.99$ each, what is the total cost?**

   $\mathbf{\$21.97}$

   b. **If Sophia purchases n songs for $\$0.99$ each, write an expression for the total cost.**

   $\mathbf{0.99n + 19.99}$

   c. **Sophia's friend has saved $\$118$ but isn't sure how many songs she can afford if she buys the membership and some songs. Use the expression in part (b) to write an equation that can be used to determine how many songs Sophia's friend can buy.**

   $\mathbf{0.99n + 19.99 = 118}$

   d. **Using the equation written in part (c), can Sophia's friend buy 101, 100, or 99 songs?**

   $\mathbf{n = 99}$

   $\mathbf{0.99n + 19.99 = 118}$

   True

   $\mathbf{n = 100}$

   $\mathbf{0.99n + 19.99 = 118}$

   $\mathbf{0.99(100) + 19.99 = 118}$

   $\mathbf{99 + 19.99 = 118}$

   $\mathbf{118.99 = 118}$

   False

   $\mathbf{n = 101}$

   $\mathbf{0.99n + 19.99 = 118}$

   $\mathbf{99.99 + 19.99 = 118}$

   False

Grade 7, Module 3, Lesson 9: Using If-Then Moves in Solving Equations

Example 1

Lead students through the following problem.

Fred and Sam are a team in the local 138.2 mile bike-run-athon. Fred will compete in the bike race and Sam will compete in the run. Fred bikes at an average speed of 8 miles per hour and Sam runs at an average speed of 4 miles per hour. The bike race begins at 6:00 a.m., followed by the run. Sam predicts he will finish the run at 2:33 a.m. the next morning.

a. **How many hours will it take for them to complete the entire bike-run-athon?**

From 6:00 a.m. to 2:00 a.m. the following day is 20 hours.

33 minutes in hours is $\frac{33}{60} = \frac{11}{20} = 0.55$ hours.

Therefore, the total time it will take to complete the entire bike-run-athon is 20.55 hours.

b. **If t is how long it takes for Fred to complete the bike race, in hours, write an expression to find Fred's total distance.**

$$d = rt$$

The expression of Fred's total distance is $8t$.

c. **Write an expression, in terms of t to express Sam's time.**

Since t is Fred's time and 20.55 is the total time, then Sam's time would be the difference between the total time and Fred's time. The expression would be $20.55 - t$.

d. **Write an expression, in terms of t, that represents Sam's total distance.**

$d = rt$

$d = 4(20.55 - t)$

The expressions $4(20.55 - t)$ or $82.2 - 4t$ is Sam's total distance.

e. **Write and solve an equation using the total distance both Fred and Sam will travel.**

$$8t + 4(20.55 - t) = 138.2$$

Fred's Time: $t = 14$ *hours*

Sam's time: $20.55 - t = 20.55 - 14 = 6.55$ *hours*

f. **How far will Fred bike, and how much time will it take him to complete his leg of the race?**

$8(14) = 112$ *miles and Fred completed the bike race in 14 hours at 8:00 p.m.*

g. **How far will Sam run and how much time will it take him to complete his leg of the race?**

Type equation here.

Problem Set

2. **The sum of a number, $\frac{1}{6}$ of that number, $2\frac{1}{2}$ of that number, and 7 is $12\frac{1}{2}$. Find the number.**

Let n represent the given number.

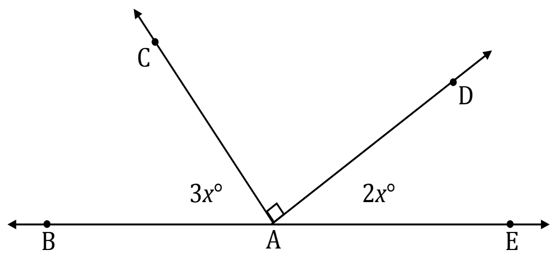
$$n + \frac{1}{6}n + \left(2\frac{1}{2}\right)n + 7 = 12\frac{1}{2}$$

$$n\left(1 + \frac{1}{6} + \frac{5}{2}\right) + 7 = 12\frac{1}{2}$$

Grade 7, Module 3, Lesson 10: Angle Problems and Solving Equations

Exercise 1

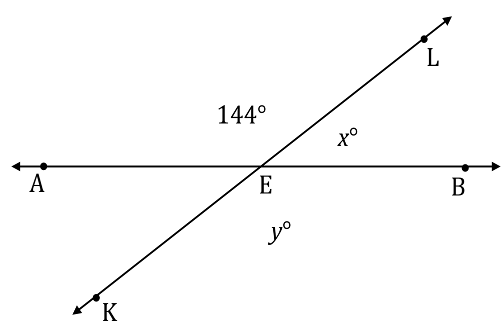
In a complete sentence, describe the angle relationship in the diagram.



Find the measurements of $\angle BAC$ and $\angle DAE$.

Example 2

In a complete sentence, describe the angle relationship in the diagram.



Write an equation for the angle relationship shown in the figure and solve for x and y. Find the measurements of $\angle LEB$ and $\angle KEB$.

Grade 8, Module 4, Lesson 4: Solving a Linear Equation

Exercises

For each problem, show your work, and check that your solution is correct.

1. Solve the linear equation: $x + x + 2 + x + 4 + x + 6 = -28$. State the property that justifies your first step and why you chose it.

2. Solve the linear equation: $2(3x + 2) = 2x - 1 + x$. State the property that justifies your first step and why you chose it.

3. Solve the linear equation: $x - 9 = \frac{3}{5}x$. State the property that justifies your first step and why you chose it.

4. Solve the linear equation: $29 - 3x = 5x + 5$. State the property that justifies your first step and why you chose it.

5. Solve the linear equation: $\frac{1}{3}x - 5 + 171 = x$. State the property that justifies your first step and why you chose it.

Grade 8, Module 4, Lesson 5: Writing and Solving Linear Equations

Example 1

One angle is five less than three times the size of another angle. Together they have a sum of 143°. What are the sizes of each angle?

Example 2

Given a right triangle, find the size of the angles if one angle is ten more than four times the other angle and the third angle is the right angle.

Grade 8, Module 4, Lesson 28: Another Computational Method of Solving a Linear System

Example 1

Use what you noticed about adding equivalent expressions to solve the following system by elimination:

$$\begin{cases} 6x - 5y = 21 \\ 2x + 5y = -5 \end{cases}$$

Example 2

Solve the following system by elimination:

$$\begin{cases} -2x + 7y = 5 \\ 4x - 2y = 14 \end{cases}$$

Grade 8, Module 5, Lesson 2:

- Suppose a moving object travels 256 feet in 4 seconds. The object is a stone, being dropped from a height of 256 feet. It takes exactly 4 seconds for the stone to hit the ground.
- In order for functions to be useful, the information we get from a function must be useful. That is why a function assigns to each input exactly one output. We also need to consider the situation when using a function. For example, if we use the function, distance for time interval $t = 16t^2$, for $t = -2$, we may be able to explain that -2 would represent 2 seconds before the stone was dropped.

  Yet, in the function

  $$\begin{aligned} \text{distance for time interval } t &= 16t^2 \quad \text{for } t = -2 \\ &= 16(-2)^2 \\ &= 16(4) \\ &= 64 \end{aligned}$$

  we could conclude that the stone dropped a distance of 64 feet two seconds before the stone was dropped! Or consider the situation when $t = 5$:

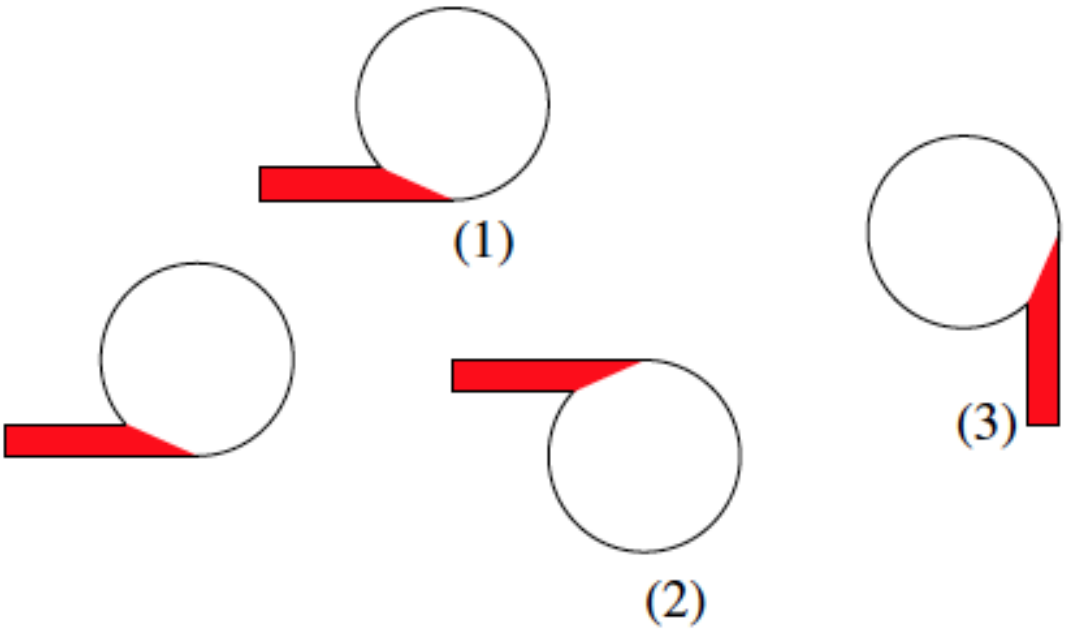
  $$\begin{aligned} \text{distance for time interval } t &= 16t^2 \quad \text{for } t = 5 \\ &= 16(5)^2 \\ &= 16(25) \\ &= 400 \end{aligned}$$

- What is wrong with this statement?
  - *It would mean that the stone dropped 400 feet in 5 seconds, but the stone was dropped from a height of 256 feet.*
- To summarize, a function is a rule that assigns to each input exactly one output. Additionally, we should always consider the context, if provided, when working with a function to make sure our answer makes sense. In many cases, functions are described by a formula. However, we will soon learn that the assignment of some functions cannot be described by a mathematical rule.

Grade 8, Module 2, Lesson 1: Why Move Things Around?

Exploratory Challenge 1

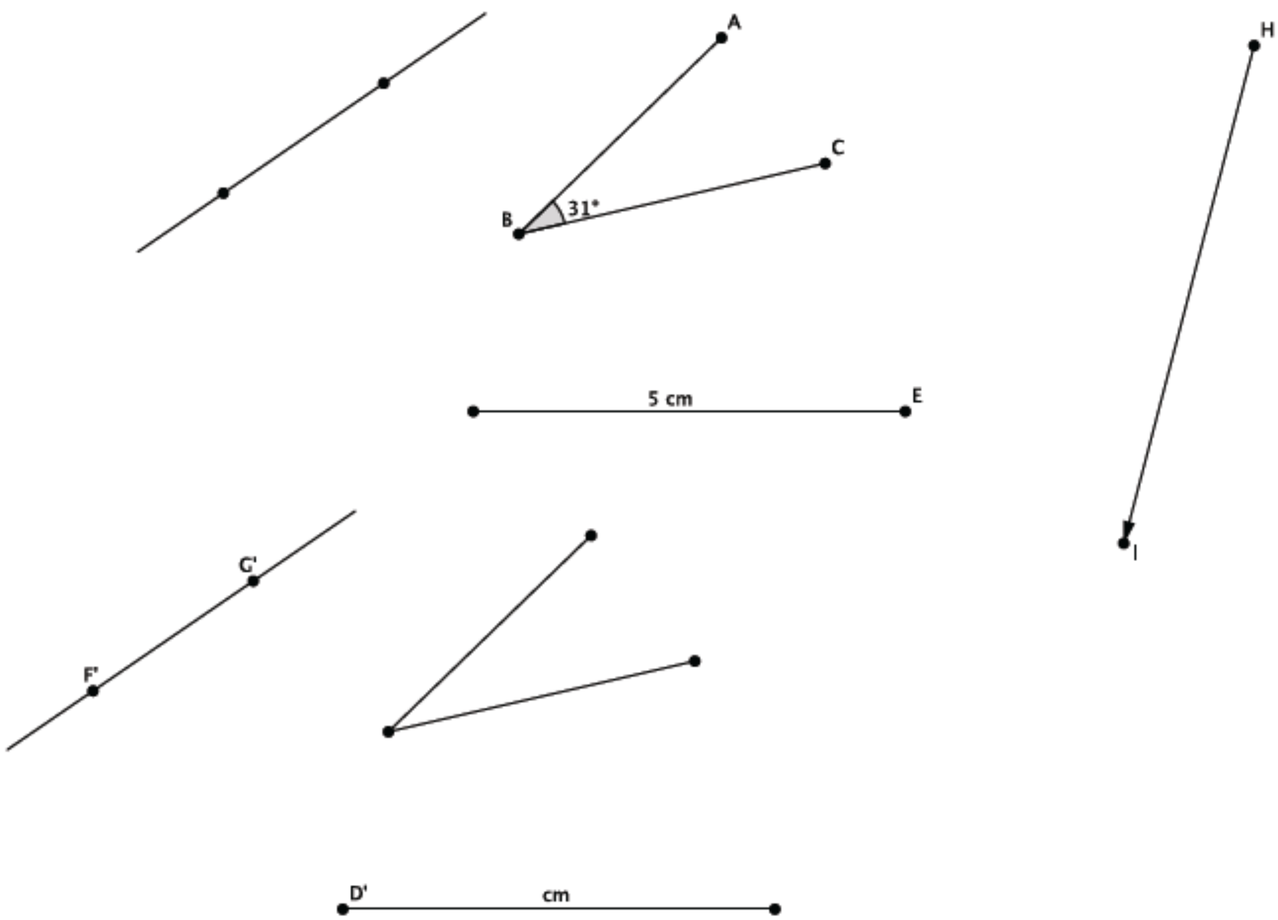
1. Describe, *intuitively*, what kind of transformation will be required to move the figure on the left to each of the figures (1 – 3) on the right. To help with this exercise, use a transparency to copy the figure on the left. Note that you are supposed to begin by moving the left figure to each of the locations in (1), (2), and (3).



Grade 8, Module 2, Lesson 2: Definition of Translation and Three Basic Properties

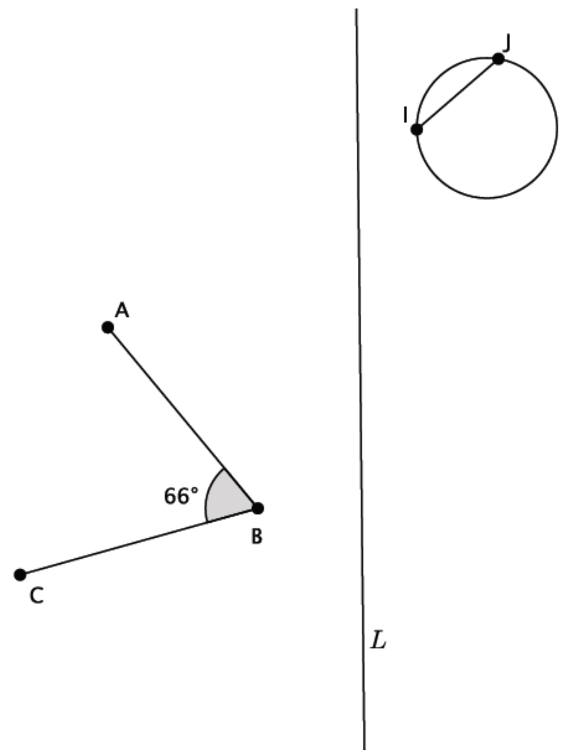
Exercise 2

The diagram below shows figures and their images under a translation along $\overrightarrow{HI}$. Use the original figures and the translated images to fill in missing labels for points and measures.



Grade 8, Module 2, Lesson 4: Definition of Reflection and Basic Properties

3. Reflect the images across line L. Label the reflected images.



4. Answer the questions about the image above.

   b. Use a protractor to measure the reflected $\angle ABC$. What do you notice?

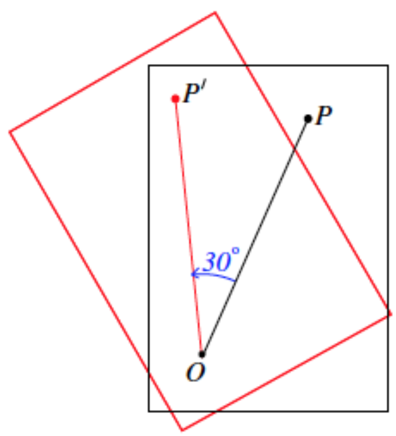
   c. Use a ruler to measure the length of IJ and the length of the image of IJ after the reflection. What do you notice?

Grade 8, Module 2, Lesson 5: Definition of Rotation and Basic Properties

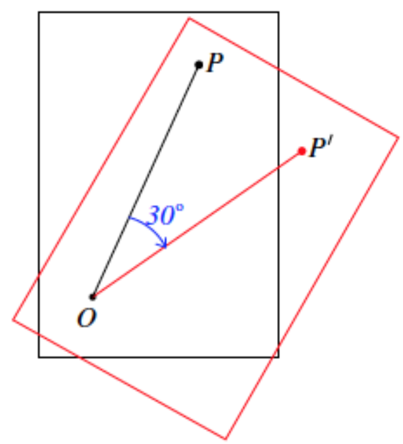
Example 1

Let there be a rotation around center O, d degrees.

If $d = 30$, then the plane moves as shown:



If $d = -30$, then the plane moves as shown:



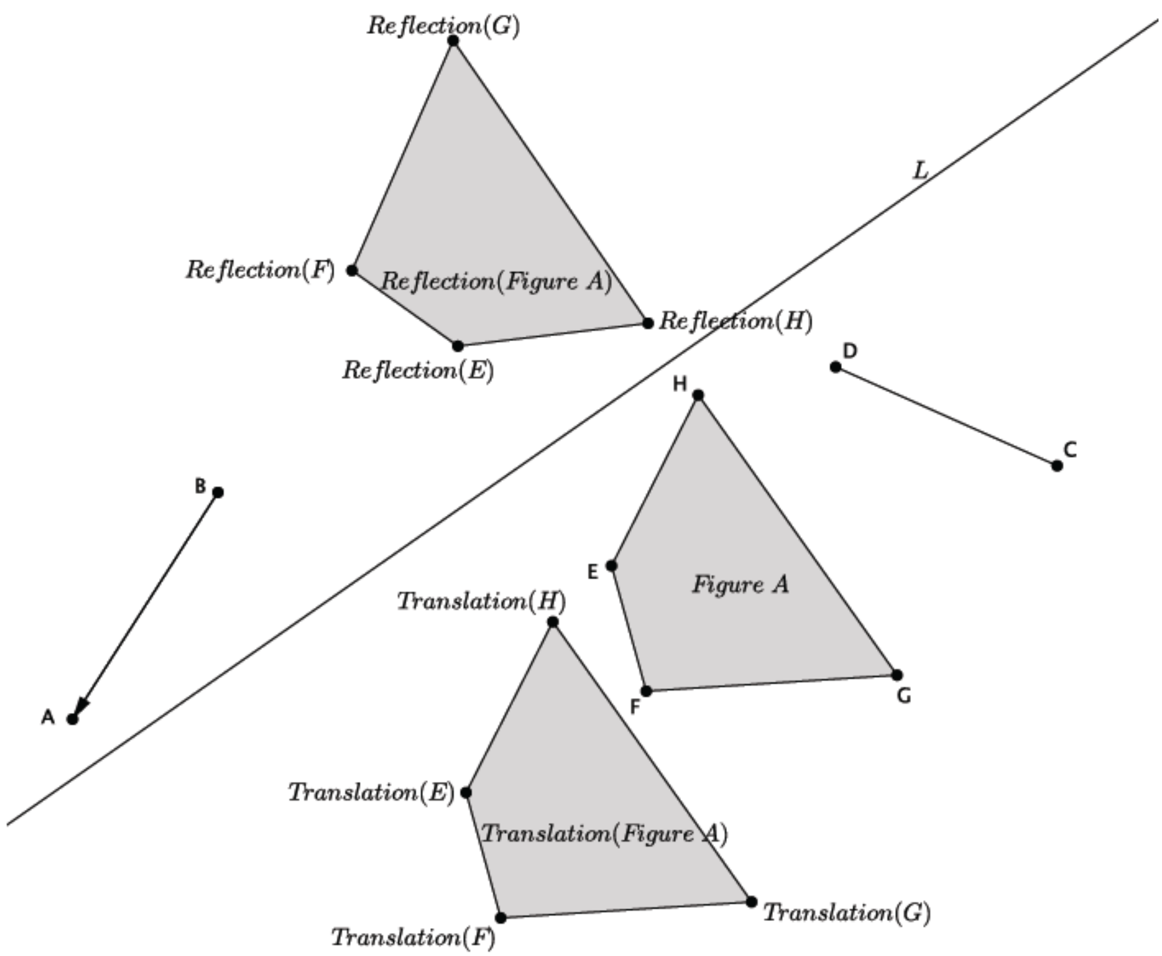
Exercise

1. Let there be a rotation of d degrees around center O. Let P be a point other than O. Select a d so that $d \geq 0$. Find P' (i.e., the rotation of point P) using a transparency.

Grade 8, Module 2, Lesson 8: Sequencing Reflections and Translations

Exercises

Use the figure below to answer Exercises $1 - 3$.

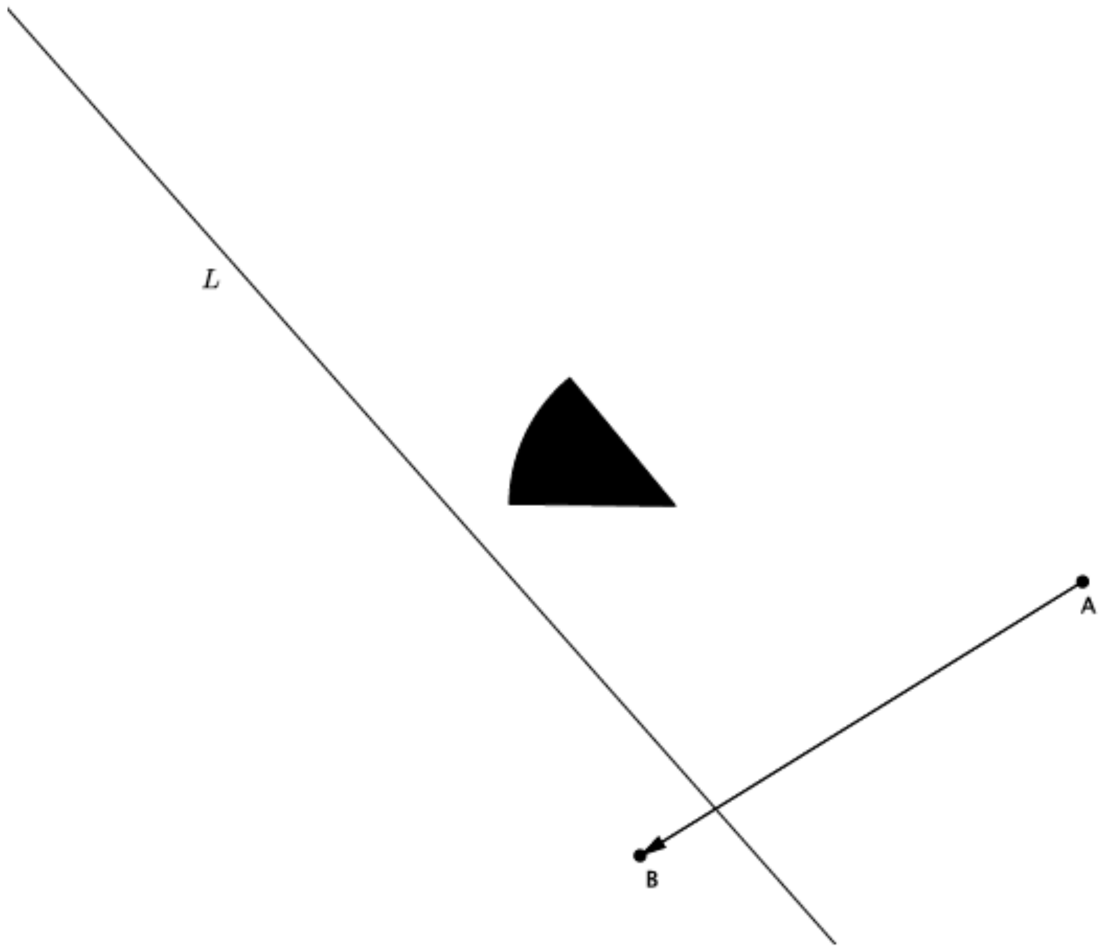


1. *Figure* A was translated along vector $\overrightarrow{BA}$ resulting in $Translation(Figure\ A)$. Describe a sequence of translations that would map *Figure* A back onto its original position.

2. *Figure* A was reflected across line L resulting in $Reflection(Figure\ A)$. Describe a sequence of reflections that would map *Figure* A back onto its original position.

3. Can $Translation_{\overrightarrow{BA}}$ undo the transformation of $Translation_{\overrightarrow{DC}}$? Why or why not?

Use the diagram below for Exercises $4 - 5$.



4. Let there be the translation along vector $\overrightarrow{AB}$ and a reflection across line L. Let the black figure above be figure S. Use a transparency to perform the following sequence: Translate figure S, then reflect figure S. Label the image S'.

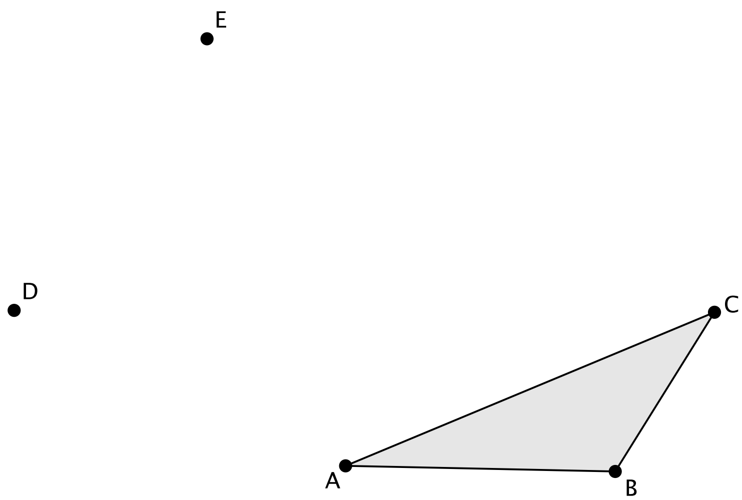
5. Let there be the translation along vector $\overrightarrow{AB}$ and a reflection across line L. Let the black figure above be figure S. Use a transparency to perform the following sequence: Reflect figure S, then translate figure S. Label the image S''

6. Using your transparency, show that under a sequence of any two translations, $Translation$ and $Translation_0$ (along different vectors), that the sequence of the $Translation$ followed by the $Translation_0$ is equal to the sequence of the $Translation_0$ followed by the $Translation.$ That is, draw a figure, A, and two vectors. Show that the translation along the first vector, followed by a translation along the second vector, places the figure in the same location as when you perform the translations in the reverse order. (This fact will be proven in high school). Label the transformed image A'. Now draw two new vectors and translate along them just as before. This time label the transformed image A''. Compare your work with a partner. Was the statement that "the sequence of the $Translation$ followed by the $Translation_0$ is equal to the sequence of the $Translation_0$ followed by the $Translation$" true in all cases? Do you think it will always be true?

7. Does the same relationship you noticed in Exercise 6 hold true when you replace one of the translations with a reflection. That is, is the following statement true: A translation followed by a reflection is equal to a reflection followed by a translation?

Grade 8, Module 2, Lesson 9: Sequencing Rotations

2.



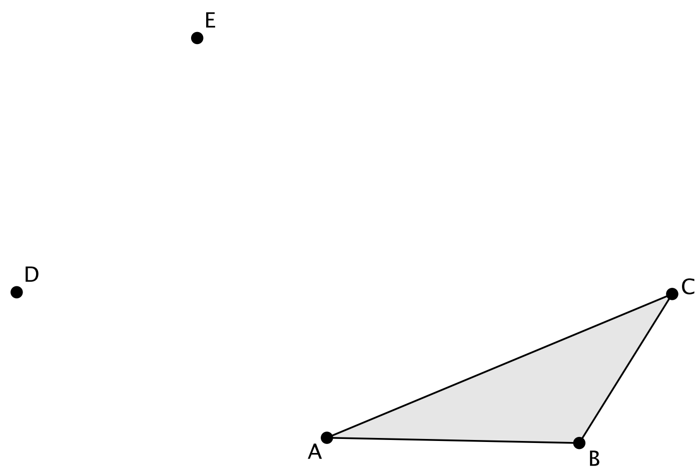
a. Rotate $\triangle ABC$ d degrees around center D and then rotate again d degrees around center E. Label the image as $\triangle A'B'C'$ after you have completed both rotations.

b. Can a single rotation around center D map $\triangle A'B'C'$ onto $\triangle ABC$?

c. Can a single rotation around center E map $\triangle A'B'C'$ onto $\triangle ABC$?

d. Can you find a center that would map $\triangle A'B'C'$ onto $\triangle ABC$ in one rotation? If so, label the center F.

3.



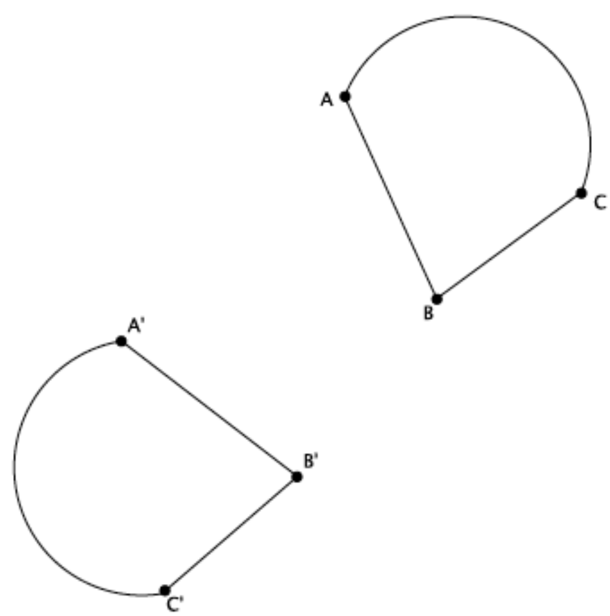
a. Rotate $\triangle ABC$ 90° (counterclockwise) around center D then rotate the image another 90° (counterclockwise around center E. Label the image $\triangle A'B'C'$.

b. Rotate $\triangle ABC$ 90° (counterclockwise) around center E then rotate the image another 90° (counterclockwise around center D. Label the image $\triangle A''B''C''$.

c. What do you notice about the locations of $\triangle A'B'C'$ and $\triangle A''B''C''$? Does the order in which you rotate a figure around different centers have an impact on the final location of the figures image?

Grade 8, Module 2, Lesson 10: Sequences of Rigid Motions

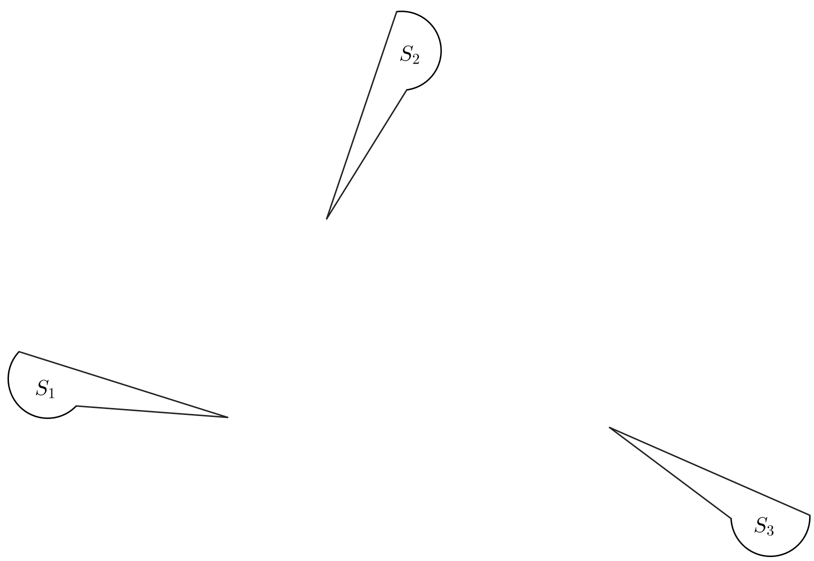
5. Let two figures ABC and $A'B'C'$ be given so that the length of curved segment $AC =$ the length of curved segment $A'C'$, $|\angle B| = |\angle B'| = 80°$, and $|AB| = |A'B'| = 5$. With clarity and precision, describe a sequence of rigid motions that would map figure ABC onto figure $A'B'C'$.



Grade 8, Module 2, Lesson 11: Definition of Congruence and Some Basic Properties

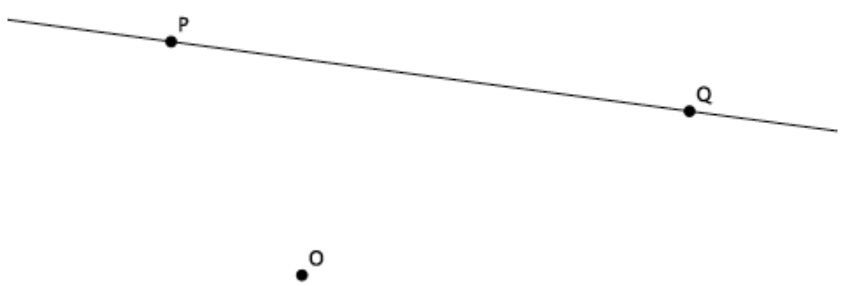
Exercise 1

a. Describe the sequence of basic rigid motions that shows $S_1 \cong S_2$.

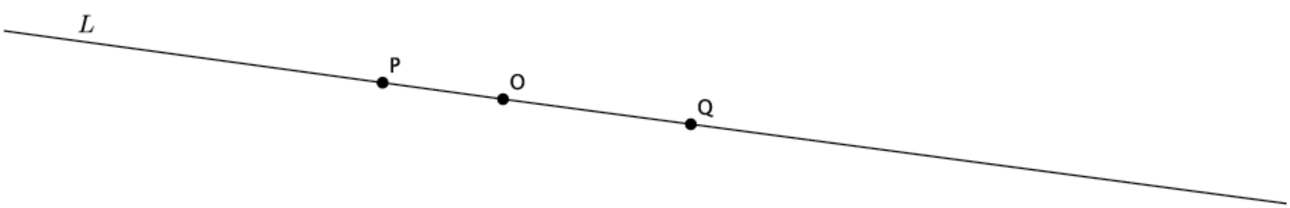


Grade 8, Module 3, Lesson 2: Properties of Dilations

Examples 1–2: Dilations Map Lines to Lines

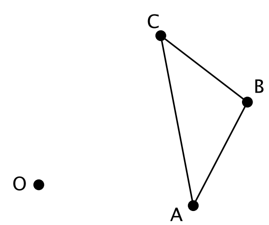


Example 3: Dilations Map Lines to Lines



Exercise

Given center O and triangle ABC, dilate the triangle from center O with a scale factor $r = 3$.



a. Note that the triangle ABC is made up of segments AB, BC, and CA. Were the dilated images of these segments still segments?

b. Measure the length of the segments AB and $A'B'$. What do you notice? (Think about the definition of dilation.)

c. Verify the claim you made in part (b) by measuring and comparing the lengths of segments BC and $B'C'$ and segments CA and $C'A'$. What does this mean in terms of the segments formed between dilated points?

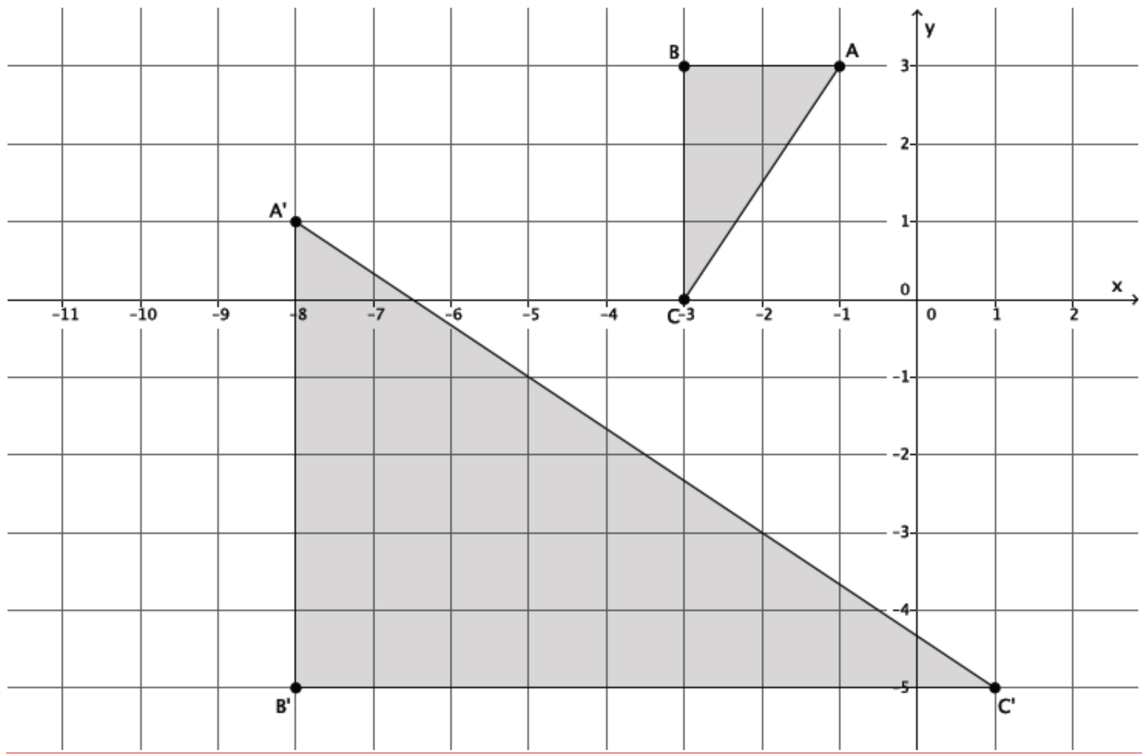
d. Measure $\angle ABC$ and $\angle A'B'C'$. What do you notice?

e. Verify the claim you made in part (d) by measuring and comparing angles $\angle BCA$ and $\angle B'C'A'$ and angles $\angle CAB$ and $\angle C'A'B'$. What does that mean in terms of dilations with respect to angles and their degrees?

Grade 8, Module 3, Lesson 8: Similarity

Exercise 2

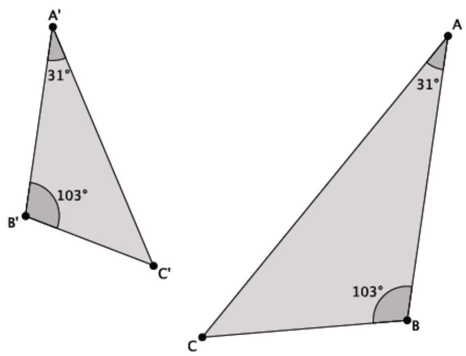
Describe the sequence that would show $\triangle ABC \sim \triangle A'B'C'$.



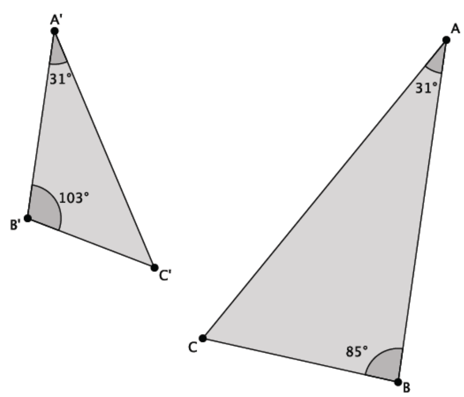
Grade 8, Module 3, Lesson 10: Informal Proof of AA Criterion for Similarity

Exercises

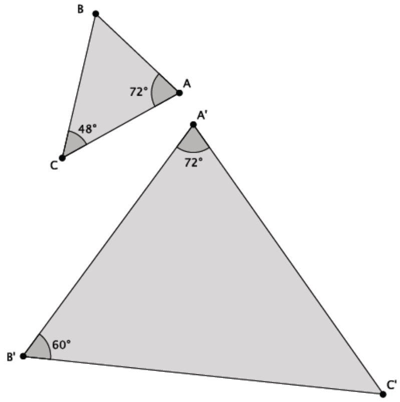
8. Are the triangles shown below similar? Present an informal argument as to why they are or why they are not.



9. Are the triangles shown below similar? Present an informal argument as to why they are or why they are not.

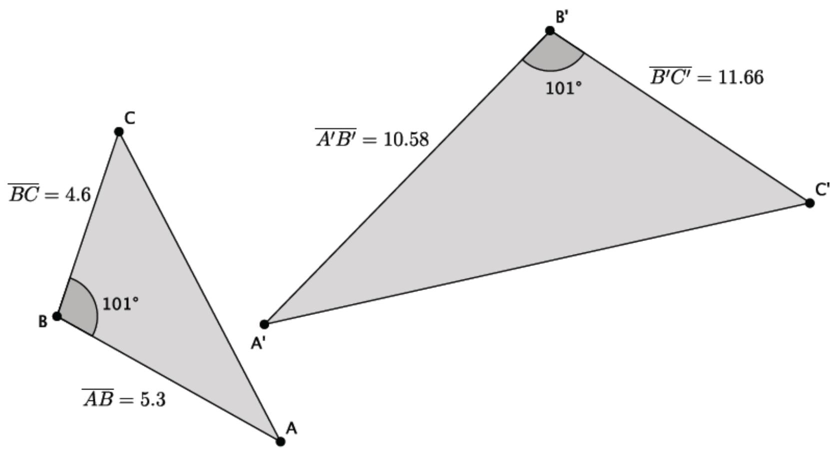


10. Are the triangles shown below similar? Present an informal argument as to why they are or why they are not.



Grade 8, Module 3, Lesson 11: More About Similar Triangles

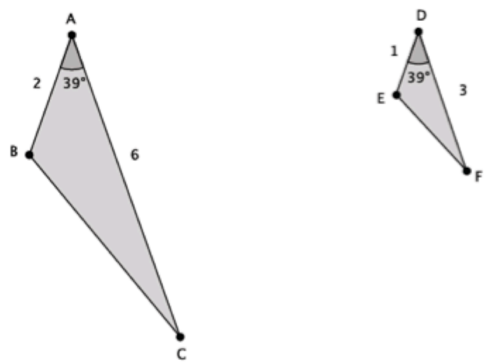
3. In the diagram below you have $\triangle ABC$ and $\triangle A'B'C'$. Use this information to answer the question below.



Based on the information given, is $\triangle ABC \sim \triangle A'B'C'$? Explain.

Lesson Summary

Given just one pair of corresponding angles of a triangle as equal, use the side lengths along the given angle to determine if triangles are in fact similar.

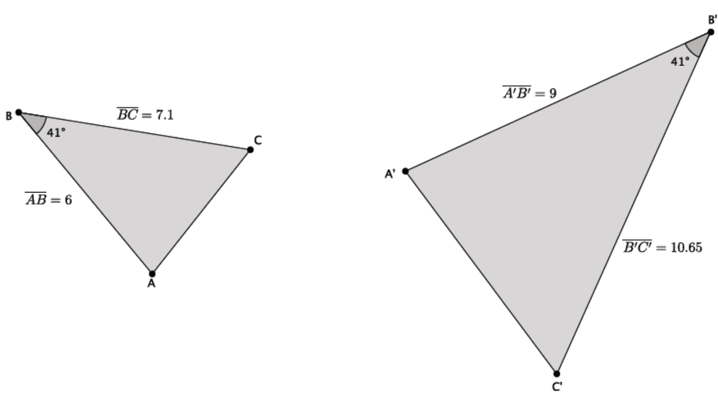


$|\angle A| = |\angle D|$ and $\frac{1}{2} = \frac{3}{6} = r$; therefore, $\triangle ABC \sim \triangle DEF$.

Given similar triangles, use the fact that ratios of corresponding sides are equal to find any missing measurements.

Problem Set

1. In the diagram below, you have $\triangle ABC$ and $\triangle A'B'C'$. Use this information to answer parts (a)–(b).



b. Based on the information given, is $\triangle ABC \sim \triangle A'B'C'$? Explain.

c. Assume the length of AC is 4.3. What is the length of $A'C'$?

Grade 8, Module 5, Lesson 1: The Concept of a Function

Example 1

MP.1 This example is used to point out that in much of our previous work, we assumed a constant rate. This is in contrast to the next example, where constant rate cannot be assumed. Encourage students to make sense of the problem and attempt to solve it on their own. The goal is for students to develop a sense of what predicting means in this context.

Suppose a moving object travels 256 feet in 4 seconds. Assume that the object travels at a constant speed, that is, the motion of the object is linear with a constant rate of change. Write a linear equation in two variables to represent the situation, and use it to make predictions about the distance traveled over various intervals of time.

Number of seconds (x)	Distance traveled in feet (y)
1	64
2	128
3	192
4	256

- Suppose a moving object travels 256 feet in 4 seconds. Assume that the object travels at a constant speed, that is, the motion of the object is linear with a constant rate of change. Write a linear equation in two variables to represent the situation, and use it to make predictions about the distance traveled over various intervals of time.
  - *Let y represent the distance traveled and x represent the time it takes to travel y feet.*

$$\frac{256}{4} = \frac{y}{x}$$
$$y = \frac{256}{4}x$$
$$y = 64x$$

- What are some of the predictions that this equation allows us to make?
  - *After one second, or when $x = 1$, the distance traveled is 64 feet.*

Accept any reasonable predictions that the students make.

- Use your equation to complete the table.
- What is the average speed of the moving object from zero to three seconds?
  - *The average speed is 64 feet per second. We know that the object has a constant rate of change; therefore, we expect the average speed to be the same over any time interval.*

Example 2

- We have already made predictions about the location of a moving object. Now, here is some more information. The object is a stone, being dropped from a height of 256 feet. It takes exactly 4 seconds for the stone to hit the ground. How far does the stone drop in the first 3 seconds? What about the last 3 seconds? Can we assume constant speed in this situation? That is, can this situation be expressed using a linear equation?

The object, a stone, is dropped from a height of 256 feet. It takes exactly 4 seconds for the stone to hit the ground. How far does the stone drop in the first 3 seconds? What about the last 3 seconds? Can we assume constant speed in this situation? That is, can this situation be expressed using a linear equation?

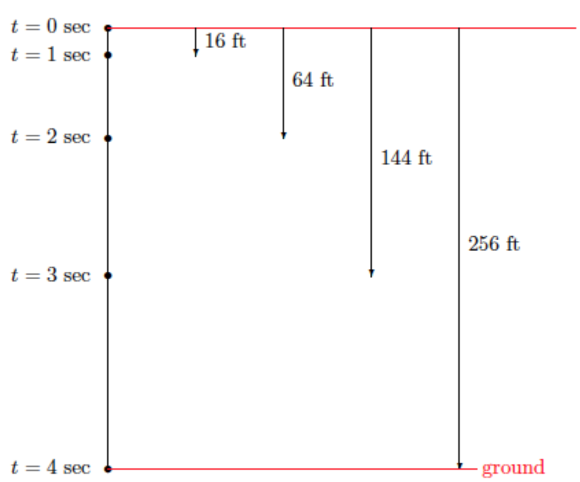
Number of seconds (x)	Distance traveled in feet (y)
1	16
2	64
3	144
4	256

Provide students time to discuss this in pairs. Lead a discussion where students share their thoughts with the class. It is likely that they will say this is a situation that can be modeled with a linear equation, just like the moving object in Example 1. Continue with the discussion below.

- If this is a linear situation, then from the table we developed in Example 1 we already know the stone will drop 192 feet in any 3 second interval. That is, the stone drops 192 feet in the first 3 seconds and in the last 3 seconds.

To provide a visual aid, consider viewing the 10 second "ball drop" video at the following link: http://www.youtube.com/watch?v=KrX_zLuwOvc. You may need to show it more than once.

- If we were to slow the video down and record the distance the ball dropped after each second, we would collect the following data:



- Choose a prediction that was made about the distance traveled before we learned more about the situation. Was it accurate? How do you know?

Grade 8, Module 5, Lesson 2: Formal Definition of a Function

Problem Set

Olivia examined the table of values shown below and stated that a possible rule to describe this function could be $y = -2x + 9$. Is she correct? Explain.

Input (x)	-4	0	4	8	12	16	20	24
Output (y)	17	9	1	-7	-15	-23	-31	-39

Peter said that the set of data in part (a) describes a function, but the set of data in part (b) does not. Do you agree? Explain why or why not.

a.

Input (x)	1	2	3	4	5	6	7	8
Output (y)	8	10	32	6	10	27	156	4

b.

Input (x)	-6	-15	-9	-3	-2	-3	8	9
Output (y)	0	-6	8	14	1	2	11	41

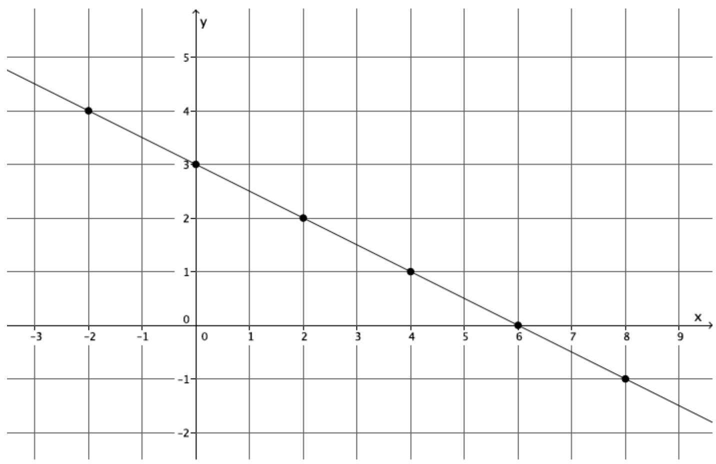
A function can be described by the rule $y = x^2 + 4$. Determine the corresponding output for each given input.

Input (x)	-3	-2	-1	0	1	2	3	4
Output (y)								

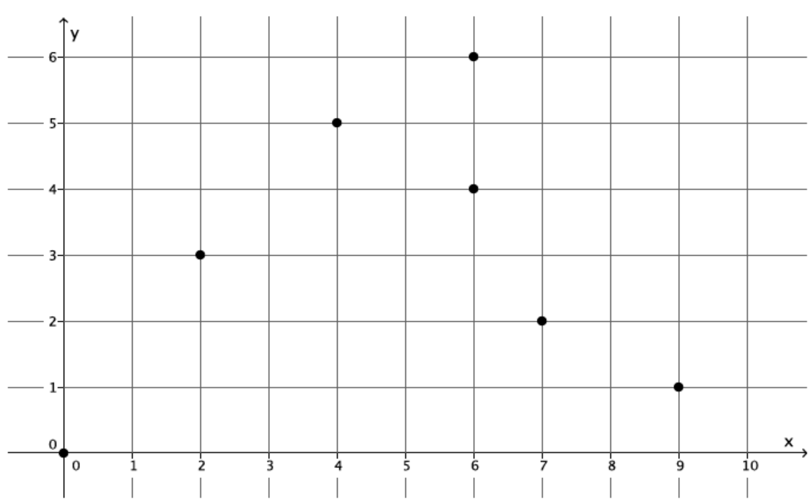
Grade 8, Module 5, Lesson 5: Graphs of Functions and Equations

4. Examine the three graphs below. Which, if any, could represent the graph of a function? Explain why or why not for each graph.

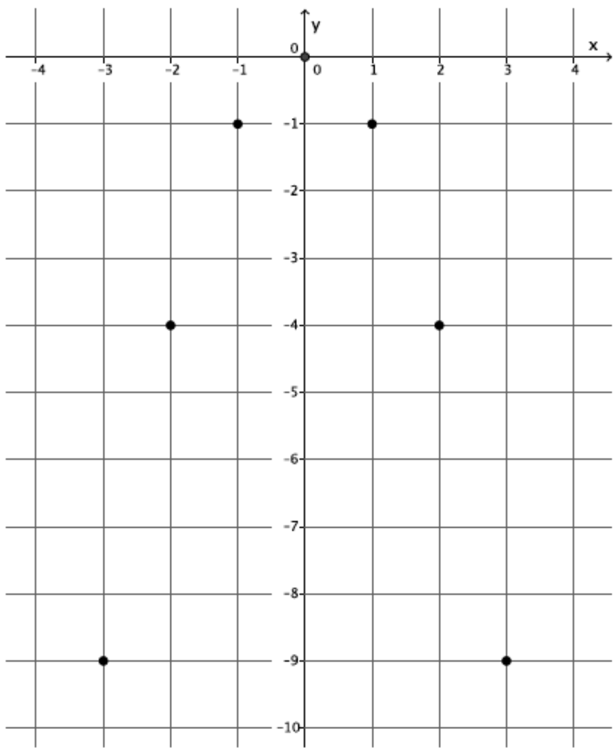
Graph 1:



Graph 2:



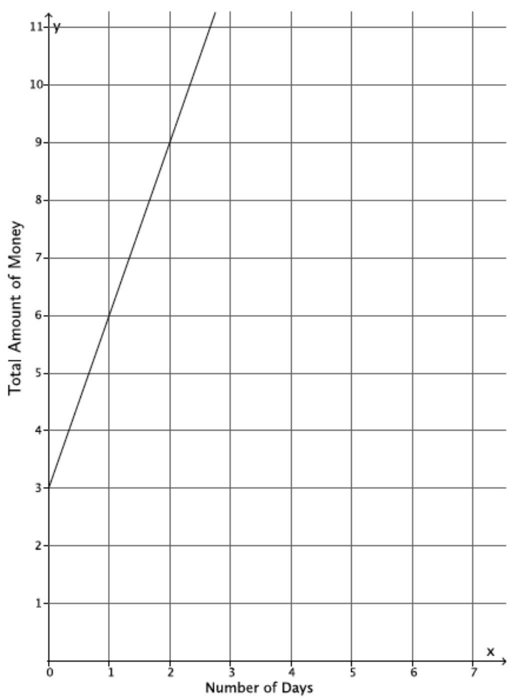
Graph 3:



Grade 8, Module 5, Lesson 7: Comparing Linear Functions and Graphs

4. Two people, Adam and Bianca, are competing to see who can save the most money in one month. Use the table and the graph below to determine who will save more money at the end of the month. State how much money each person had at the start of the competition.

Adam's Savings:



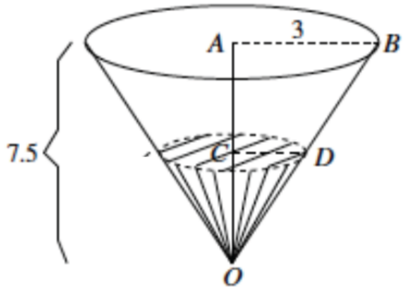
Bianca's Savings:

Input (Number of Days)	Output (Total amount of money)
5	$\$17$
8	$\$26$
12	$\$38$
20	$\$62$

Grade 8, Module 7, Lesson 22: Average Rate of Change

Discussion (30 minutes)

The height of a container in the shape of a circular cone is 7.5 ft. and the radius of its base is 3 ft., as shown. What is the total volume of the cone?



The volume of the cone is

$$V = \frac{1}{3}\pi 3^2(7.5) = 22.5\pi$$

- If we knew the rate at which the cone was being filled with water, how could we use that information to determine how long it would take to fill the cone?
  - *We could take the total volume and divide it by the rate to determine how long it would take to fill.*
- Water flows into the container (in its inverted position) at a constant rate of 6 ft^3 per minute. Approximately when will the container be filled?

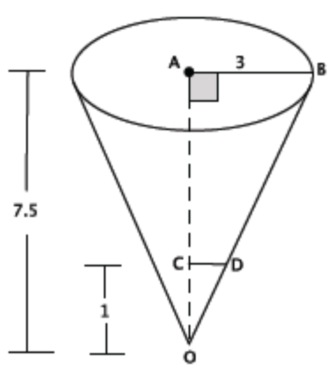
Provide students time to work in pairs on the problem. Have students share their work and their reasoning about the problem.

  - *Since the container is being filled at a constant rate then the volume must be divided by the rate at which it is being filled (using* 3.14 *for* π *and rounding to the hundredths place):*

$$\frac{22.5\pi}{6} \approx 11.78\ min$$

It will take almost 12 *minutes to fill the cone at a rate of* 6 ft^3 *per minute.*

- Now we want to show that even though the water filling the cone flows at a constant rate, the rate of change of the volume in the cone is not constant. For example, if we wanted to know how many minutes it would take for the level in the cone to reach 1 ft., then we would have to first determine the volume of the cone when the height is 1 ft. Do we have enough information to do that?
  - *Yes, we will need to first determine the radius of the cone when the height is* 1 *ft.*



- What equation can we use to determine the radius when the height is 1 ft.? Explain how your equation represents the situation.
  - *If we let $|CD|$ represent the radius of the cone when the height is 1 ft., then*

$$\frac{3}{|CD|} = \frac{7.5}{1}$$

The number 3 represents the radius of the original cone. The 7.5 represents the height of the original cone, and the 1 represents the height of the cone we are trying to solve for.

- Use your equation to determine the radius of the cone when the height is 1 ft.
  - *The radius when the height is 1 ft. is*

$$3 = 7.5|CD|$$
$$\frac{3}{7.5} = |CD|$$
$$0.4 = |CD|$$

- Now determine the volume of the cone when the height is 1 ft.
  - *Then we can find the volume of the cone with a height of 1 ft.:*

$$V = \frac{1}{3}\pi(0.4)^2(1)$$
$$= \frac{0.16}{3}\pi$$

Now we can divide the volume by the rate at which the cone is being filled to determine how many minutes it would take to fill a cone with a height of 1 ft.:

$$\frac{0.16}{3}\pi \approx 0.167$$
$$\frac{0.167}{6} \approx 0.028$$

It would take about 0.028 minutes to fill a cone with a height of 1 ft.

- Calculate the number of minutes it would take to fill the cone at 1 ft. intervals. Organize your data in the table below.

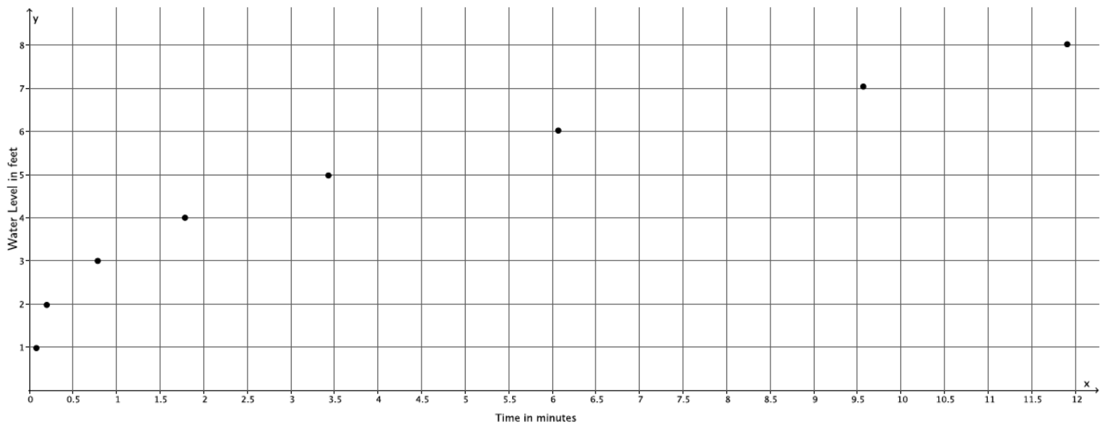
Provide students time to work on completing the table. They should replicate the work above by first finding the radius of the cone at the given heights, then using the radius to determine the volume of the cone, and then the time it would take to fill a cone of that volume at the given constant rate. Once most students have finished, continue with the discussion below.

Time in minutes	Water level in feet
0.028	1
0.22	2
0.75	3
1.78	4
3.49	5
6.03	6
9.57	7
11.78	7.5

MP.3

- We know that the sand (rice, water, etc.) being poured into the cone is poured at a constant rate, but is the level of the substance in the cone rising at a constant rate? Provide evidence to support your answer.

Provide students time to construct an argument based on the data collected to show that the substance in the cone is not rising at a constant rate. Have students share their reasoning with the class. Students should be able to show that the rate of change (slope) between any two data points is not the same using calculations like $\frac{2-1}{0.22-0.028}=\frac{1}{0.192}=5.2$ and $\frac{7-6}{9.57-6.03}=\frac{1}{3.54}=0.28$ or by graphing the data and showing that it is not linear.



Close the discussion by reminding students of the demonstration at the Opening of the lesson. Ask students if the math supported their conjectures about average rate of change of the water level of the cone.

Closing (5 minutes)

Consider asking students to write a summary of what they learned. Prompt them to include a comparison of how filling a cone is different from filling a cylinder. Another option is to have a whole class discussion where you ask students how to interpret this information in a real-world context. For example, if they were filling a cylindrical container and a conical container with the same radius and height, which would fill first? Or the example, would the rate of change of the volume be different if we were emptying the cone as opposed to filling it? Would the rate of change in the water level be different if we were emptying the cone as opposed to filling it? How so? What might that look like on a graph?

Summarize, or ask students to summarize, the main points from the lesson:

- We know intuitively that the narrower part of a cone will fill up faster than the wider part of a cone.
- By comparing the time it takes for a cone to be filled to a certain water level, we can determine that the rate of filling the cone is not constant.